On $\sigma-$additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model

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Abstract

Recently, Frittelli and Scandolo ([7]) extend the notion of risk measures, originally introduced by Artzner, Delbaen, Eber and Heath ([1]), to the risk assessment of abstract financial positions, including pay offs spread over different dates, where liquid derivatives are admitted as financial instruments, and unbounded fincial positions are also allowed. Convex risk measures may be viewed as convex upper previsions for unbounded gambles, a notion originally introduced by Pelessoni and Vicig [16]. The paper deals with $\sigma-$additive robust representations of convex risk measure, that means envelope theorems in terms of $\sigma-$additive probability measures. We shall focus on the aspect that the investor is faced with uncertainty about the market model. It turns out that the results may be applied for the case that a market model is available, and that they encompass as well as improve criteria obtained for robust representations of convex risk measures in the genuine sense ([2], [5], [13]).

Keywords. Convex risk measures, convex upper previsions, model uncertainty, $\sigma-$additive robust representation, Greco's representation theorem, Fatou property, inner Daniell stone theorem, general Dini theorem, strong $\sigma-$additive robust representation, Simons' lemma, nonsequential Fatou property, Krein-Smulian theorem.

1 Introduction

The notion of risk measures has been introduced by Artzner, Delbaen, Eber and Heath (cf. [1]) as the key concept to found an axiomatic approach for risk assessment of fincancial positions. Technically, risk measures are functionals defined on sets of financial positions, satisfying some basic properties to qualify riskiness consistently. An outcome of such a functional, that means the risk of a position, is usually interpreted as the capital requirement of the position to become an acceptable one. Genuinely, risk measures has been defined for one-period positions. Recently Fritelli and Scandolo ([7]) provide a general framework which extends considerations to abstract financial positions including pay off streams with liquid derivatives as hedging positions. Applied to the risk assessment of pay off streams such general risk measures are used for an a priori qualification, which means to take the static perspective. In contrary the dynamic risk assessment take into account adjustments time after time. Readers who interested in this topic are referred to e.g. [6], [17], [21].

The main goal of this paper is to investigate risk measures ρ which admit a robust representation of the form

$$\rho(X) = \sup_{\Lambda}(-\Lambda(X) - \beta(\Lambda)),$$

where X denotes a financial position, Λ a linear form on the set of financial positions, and β stands for a penalty function on the set of linear forms. Special attention will be paid to the problem when these representing linear forms may in turn be represented by ($\sigma-$additive) probability measures. We shall speak of a robust representation of ρ by probability measures or a $\sigma-$additive robust representation. Necessarily, only so-called convex risk measures, that means risk measures which are convex mappings, may have such a robust representation. The basic assumption of this paper is that the investors are uncertain about the market model underlying the outcomes of the financial positions. Within this setting a robust representation by probability measures offered an additional economic interpretation of the risk measures. As suggested by Föllmer and Schied (cf. [5]) such a representation means that an investor has a set of possible market models in mind, and evaluates the worst expected losses together with some penalty costs for misspecification w.r.t. these models. In particular an investor with such a risk measure may be viewed as

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risk- and ambiguity-averse (cf. [19]).

The problem of $\sigma-$additive robust representation of convex risk measures in the genuine sense has been completely solved in the case that the investors have market models at hand. Ruszczynski and Shapiro showed that convex risk measures always admit robust representations by probability measures if for any real p every integrable mapping of order p is available (cf. [18]). However the used methods can not be applied to essentially bounded positions. Drawing on methods from functional analysis, Delbaen as well as Föllmer and Schied succeeded in giving a full characterization (cf. [2], [5]) by the so-called Fatou property. As pointed out by Delbaen, the Fatou property fails to be sufficient in general when the investor is faced with model uncertainty. Moreover, the problem of $\sigma-$additive robust representation is still open when a market model is not available. Restricting considerations to bounded one-period positions, Föllmer and Schied suggested a strict sufficient criterion, Krätschmer showed that it is in some sense also necessary, and he adds some more general conditions ([13]).

This paper may be viewed as a continuation of the studies by in [5] as well as in [13]. The generalizations will be proceeded into several directions. First of all multiperiod positions and liquid hedging instruments will be allowed. Secondly we shall drop the assumptions that only bounded positions are traded. This is in accordance with empirical evidences that the distributions of risky assets often show heavy tails. Thirdly we want to investigate the issue of strong robust representations by probability measures in the sense that the optimization involved in the $\sigma-$additive robust representation has a solution. Finally, the criteria should encompass the results already derived within a fixed market model.

The paper is organized as follows. Section 2 introduces the concept of Frittelli and Scandolo to define risk measures in general, and some representation results of risk measures will be presented as starting points for the following investigations. The general criterion is offered in section 3, extending a former result in [13] to unbounded positions, within a nontopological framework. It will be used for strong robust representations of risk measures by probability measures in section 4. We shall succeed in giving a complete solution. In particular the aboved mentioned strict criterion by Föllmer and Schied will turn out to be necessary and sufficient. Moreover, within a given market model the solution by Jouini, Schachermayer and Touzi (in [8]) may be recognized. Afterwards, section 5 deals with the question when the Fatou property might be used as a sufficient condition. In presence of a market model the results may be used to retain the above mentioned equivalent characterization by Delbaen as well as Föllmer and Schied. In general, as a rule a nonsequential counterpart is more suitable unless in some special cases.

The proofs of the results presented within this paper use several arguments from functional analyis, particularly from convex and superconvex analysis, as well as from abstract measure and integration theory. They are very technical in nature and must be omitted due to limitations of scope. The interested reader is kindly referred to the working paper version [15].

2 Some basic representations of convex risk measures

Let us fix a set Ω. Financial positions will be expressed by mappings $X \in \mathbb{R}^{\Omega}$. As a special case $\Omega = \widetilde{\Omega} \times \mathbb{T}$ with $\widetilde{\Omega}$ denoting a set of scenarios, equipped with a family $(\mathcal{F}_t)_{t\in\mathbb{T}}$ of $\sigma-$algebras, and $\mathbb{T}$ being a time set, we may consider financial positions $X \in \mathbb{R}^{\widetilde{\Omega}\times\mathbb{T}}$ with $X(\cdot, t)$ being $\mathcal{F}_t-$measurable for every $t \in \mathbb{T}$. They may be viewed as discounted pay off streams, liquidated at the dates from the time set. In the case of $\mathbb{T} = \{1\}$ we shall speak of **one-period positions**. The available financial positions are gathered by a nonvoid vector subspace $\mathfrak{X} \subseteq \mathbb{R}^{\Omega}$ containing the constants. Sometimes we shall in addition assume that $X \wedge Y := \min\{X, Y\}, X \vee Y := \max\{X, Y\} \in \mathfrak{X}$ for $X, Y \in \mathfrak{X}$. In this case $\mathfrak{X}$ is a so-called Stonean vector lattice. For the space of bounded positions from $\mathfrak{X}$ the symbol $\mathfrak{X}_b$ will be used. Furthermore let us fix a vector subspace $\mathfrak{C} \subseteq \mathfrak{X}$ of financial positions for hedging, including the constants. This means that we may take into account liquid derivatives like put and call options as financial instruments. In particular, in the case of pay off streams we may also allow investments and disinvestments varying over the time. The financial positions are associated with a positive linear function $\pi : \mathfrak{C} \to \mathbb{R},\ \pi(1) = 1$, where $\pi(Y)$ stands for the initial costs to obtain Y. In the seminal paper by Artzner et al. in [1] considerations are restricted to one-period positions and π being the identity on $\mathbb{R}$. Let us now introduce the concept of risk measures suggested by Frittelli and Scandolo in [7]. As for one-period positions we may choose the axiomatic viewpoint, defining a **risk measure w.r.t.** π to be a functional $\rho : \mathfrak{X} \to \mathbb{R}$ which satisfies the properties

- **monotonicity:**
  $\rho(X) \leq \rho(Y)$ for $X \geq Y$
- **translation invariance w.r.t.** π**:**
  $\rho(X + Y) = \rho(X) - \pi(Y)$ for $X \in \mathfrak{X}, Y \in \mathfrak{C}$

The meaning of these conditions may be transferred from the genuine concept of risk measures. Moreover, it can be shown that a risk measure ρ w.r.t. π satisfies $\rho(X) = \inf\{\pi(Y) \mid Y \in \mathfrak{C}, \rho(X+Y) \leq 0\}$ for any $X \in \mathfrak{X}$ ([7], Proposition 3.6). Regarding $\rho^{-1}(]-\infty, 0])$ as the acceptable positions, an outcome $\rho(X)$ expresses the infimal costs to hedge it. This retains the original meaning of risk measures as capital requirements.

In the following we shall focus on so-called **convex risk measures**, defined to mean risk measures which are convex mappings. Convexity is a reasonable condition for a risk measure due to its interpretation that diversification should not increase risk. From the technical point of view convexity is a necessary property for the desired dual representations of risk measures. Convex risk measures may be viewed as convex upper previsions as introduced in [16]. More precisely, if $\bar{\mathrm{P}}$ denotes a convex upper prevision on the gambles from $\mathfrak{X}$, then ρ defined by $\rho(X) := \bar{\mathrm{P}}(-X)$ is a convex risk measure w.r.t. the identity on $\mathbb{R}$.

Let us now fix a convex risk measure $\rho : \mathfrak{X} \to \mathbb{R}$ w.r.t. π. It is associated with $\beta_\rho : \mathfrak{X}^* \to]-\infty, \infty]$, defined by

$$\beta_\rho(\Lambda) = \sup_{X \in \mathfrak{X}} (-\Lambda(X) - \rho(X)) = \rho^*(-\Lambda),$$

where $\mathfrak{X}^*$ gathers all real linear forms on $\mathfrak{X}$, and ρ^* denotes the so-called Fenchel-Legendre transform of ρ. It is easy to verify that every Λ from the domain $\beta_\rho^{-1}(\mathbb{R})$ of β_ρ has to be a positive linear form extending π. The standard tools from convex analysis provide basic representation results for ρ with β_ρ as a penalty function.

Proposition 1 *Let $\mathfrak{X}_+^{*\pi}$ denote the space of all positive linear forms on $\mathfrak{X}$ which extend π, and let τ be any topology on $\mathfrak{X}$ such that $(\mathfrak{X}, \tau)$ is a locally convex topological vector space with topological dual $\mathfrak{X}'$. Then $\rho(X) = \max_{\Lambda \in \mathfrak{X}_+^{*\pi}} (-\Lambda(X) - \beta_\rho(\Lambda))$ for every $X \in \mathfrak{X}$. Moreover, $\rho(X) = \sup_{\Lambda \in \mathfrak{X}_+^{*\pi} \cap \mathfrak{X}'} (-\Lambda(X) - \beta_\rho(\Lambda))$ holds for every $X \in \mathfrak{X}$ if and only if ρ is lower semicontinuous w.r.t. τ.*

The proof may be found in [15] (AppendixB).

The aim of the paper is to improve the representation results by allowing only representing linear forms which are in turn representable by $\sigma-$additive probability measures. For notational purposes let us introduce the counterpart of β_ρ w.r.t. the probability measures on the $\sigma-$algebra $\sigma(\mathfrak{X})$ on Ω generated by $\mathfrak{X}$

$$\alpha_\rho : \mathcal{M}_1 \to]-\infty, \infty], \ \mathrm{P} \mapsto \sup_{X \in \mathfrak{X}} (-E_{\mathrm{P}}[X] - \rho(X)).$$

Here $\mathcal{M}_1$ is defined to consist of all $\sigma-$additive probability measures P on $\sigma(\mathfrak{X})$ such that all positions from $\mathfrak{X}$ are P $-$integrable, and $E_{\mathrm{P}}[X]$ denotes the expected value of X w.r.t. P . We shall speak of a **robust representation by probability measures from** $\mathcal{M}$ or a $\sigma-$**additive robust representation of** ρ **w.r.t.** $\mathcal{M}$ if $\mathcal{M} \subseteq \mathcal{M}_1$ nonvoid, and the representation $\rho(X) = \sup_{\mathrm{P} \in \mathcal{M}} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ holds for every $X \in \mathfrak{X}$. As an immediate consequence of Proposition 1 we obtain a first characterization of such representations.

Proposition 2 *Let F be a vector space of bounded countably additive set functions on $\sigma(\mathfrak{X})$ which separates points in $\mathfrak{X}$ such that each $X \in \mathfrak{X}$ is integrable w.r.t. any $\mu \in F$. Then in the case that the set $\mathcal{M}_1(F)$ of all $\mathrm{P} \in \mathcal{M}_1 \cap F$ with $E_{\mathrm{P}}|\mathfrak{C} = \pi$ is nonvoid*

$$\rho(X) = \sup_{\mathrm{P} \in \mathcal{M}_1(F)} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P})) \text{ for all } X \in \mathfrak{X}$$

if and only if ρ is lower semicontinuous w.r.t. weak topology $\sigma(\mathfrak{X}, F)$ on $\mathfrak{X}$ induced by F.

Remark 1 *Retaking assumptions and notations from Proposition 2, ρ admits a robust representation in terms of $\mathcal{M}_1(F)$ if F contains the Dirac measures, and if $\liminf_i \rho(X_i) \geq \rho(X)$ holds for every net $(X_i)_{i \in I}$ in $\mathfrak{X}$ which converges pointwise to some X from $\mathfrak{X}$.*

In general the lower semicontinuity of ρ w.r.t. the topology from Proposition 2 is not easy to verify. Therefore we are looking for more accessible conditions. The considerations will be based on the crucial step to reduce the investigations to bounded financial positions. That means ρ should admit a $\sigma-$additive robust representation if and only if the restriction to the bounded positions does so. In the case that $\mathfrak{X}$ is in addition a Stonean vector lattice this may be achieved via Greco's representation theorem (cf. [11], Theorem 2.10 with Remark 2.3) if the linear forms from the domain of β_ρ are representable as asymmetric Choquet integrals w.r.t. a finitely additive probability measure (cf. [15], Lemma 6.5). The reader may consult the monograph [4] for the concept of asymmetric Choquet integrals w.r.t. isotone set functions. Fortunately, drawing on Greco's representation theorem again, we might express this condition equivalently by the property that the **cutting condition** $\lim_{n \to \infty} \rho(-\lambda(X-n)^+) = \rho(0)$ $((X-n)^+ := (X-n) \vee 0)$ is satisfied for every $\lambda > 0$ and any nonnegative $X \in \mathfrak{X}$ (cf. [15], Proposition 6.6). To summarize

Proposition 3 *Let $\mathfrak{X}$ be a Stonean vector lattice, and let $\lim_{n \to \infty} \rho(-\lambda(X-n)^+) = \rho(0)$ be fulfilled for every*

$\lambda > 0$ *and every nonnegative* $X \in \mathfrak{X}$. *Then for any nonvoid* $\mathcal{M} \subseteq \mathcal{M}_1$ *the following statements are equivalent*

.1 $\rho(X) = \sup_{Q \in \mathcal{M}} (-E_Q[X] - \alpha_\rho(Q))$ *for all bounded* $X \in \mathfrak{X}$

.2 $\rho(X) = \sup_{Q \in \mathcal{M}} (-E_Q[X] - \alpha_\rho(Q))$ *for all* $X \in \mathfrak{X}$.

The cutting condition will be the basic assumption for the general representation result of the paper. Essentially, it says that for a seller of a derived call option the risk of a loss tends to the risk of inactivity with increasing strike price. Notice that the cutting condition is redundant if all positions in $\mathfrak{X}$ are bounded.

Before going into the development of criteria for $\sigma-$additive representations let us collect some necessary conditions. In the case that the positions from $\mathfrak{X}$ are essentially bounded mappings w.r.t. a reference probability measure of a given market model the so-called Fatou property plays a prominent role. Adapting this concept, we shall say that a risk measure ρ fulfills the **Fatou property** if the inequality $\liminf_{n\to\infty} \rho(X_n) \geq \rho(X)$ holds whenever $(X_n)_n$ is a uniformly bounded sequence in $\mathfrak{X}$ which converges pointwise to some bounded $X \in \mathfrak{X}$. The Fatou property implies obviously that $\rho|\mathfrak{X}_b$ is **continuous from above**, defined to mean $\rho(X_n) \nearrow \rho(X)$ for $X_n \searrow X$. Both conditions coincide if $\sup_n X_n \in \mathfrak{X}$ for any uniformly bounded sequence $(X_n)_n$ in $\mathfrak{X}$.

Proposition 4 *Let* ρ *admit a* $\sigma-$*additive robust representation w.r.t. some nonvoid* $\mathcal{M} \subseteq \mathcal{M}_1$, *then* ρ *satisfies the Fatou property, and* $\rho|\mathfrak{X}_b$ *is continuous from above.*

The proof may be found in [15] (section 7).

3 Robust representation of convex risk measures by inner regular probability measures

Throughout this section let $\mathfrak{X}$ be a Stonean vector lattice, and let $\mathfrak{L} \subseteq \mathfrak{X}$ denote any Stonean vector lattice which contains $\mathfrak{C}$ as well as generates $\sigma(\mathfrak{X})$, and which induces the set system $\mathcal{S}$ consisting of all $\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[)$, where $X_n \in \mathfrak{L}$ nonnegative, bounded, $x_n > 0$. Additionally, let E consist of all bounded $\sup_n Y_n$, where $(Y_n)_n$ is a sequence of nonnegative bounded positions from $\mathfrak{L}$.

One might think of an investor who is not aware of his or her preferences on the entire space $\mathfrak{X}$ but only on the subspace $\mathfrak{L}$. Let us also assume that he or she has a class of possible market models in mind yielding a $\sigma-$additive robust representation of $\rho|\mathfrak{L}$. Then for the modelling of the preferences on the whole set $\mathfrak{X}$ of available positions it might be useful for the investor to have conditions to hand which lead to a risk assessment consistent with her or his risk- and ambiguity-aversity expressed by the $\sigma-$additive robust representation of $\rho|\mathfrak{L}$.

First of all, in view of the inner Daniell-Stone theorem (cf. [11], Theorem 5.8, final remark after Addendum 5.9) every probability measure $P \in \mathcal{M}_1$ has to be inner regular w.r.t. $\mathcal{S}$, i.e. $P(A) = \sup_{A \supseteq B \in \mathcal{S}} P(B)$ for every $A \in \sigma(\mathfrak{X})$. So within this setting we are dealing with robust representations of ρ by probability measures from $\mathcal{M}_1(\mathcal{S})$ defined to consist of all probability measures belonging to $\mathcal{M}_1$ which are inner regular w.r.t. $\mathcal{S}$ and which represent π on $\mathfrak{C}$. As a consequence we obtain the following necessary condition for a $\sigma-$additive robust representation of ρ (cf. [15], section 7).

Proposition 5 *If* ρ *has a robust representation w.r.t. some* $\mathcal{M} \subseteq \mathcal{M}_1$, *then* $\rho(X) = \sup_{X \leq Y \in E} \inf_{Y \geq Z \in \mathfrak{X}} \rho(Z)$ *for every bounded nonegative* $X \in \mathfrak{X}$.

Imposing the cutting condition, it remains to focus on the nonnegative bounded positions for representation purposes due to Proposition 3 and the translation invariance of ρ. Then by the necessary regularity from Proposition 5 the restriction of ρ to the bounded positions has to be already determined by the values of ρ at the bounded positions from $\mathcal{L}$. Moreover, a $\sigma-$additive robust representation might be guaranteed if the following properties are satisfied

(*) $\Lambda|\mathcal{L}$ is representable by a probability measure from $\mathcal{M}_1(\mathcal{S})$ for $\beta_\rho(\Lambda) < \infty$,

(**) $\alpha_\rho(P) = \sup_{Y \in \mathcal{L}} (-E_P[Y] - \rho(Y))$ for $\alpha_\rho(P) < \infty$.

Property $(*)$ means that the investor's risk assessment of the positions from $\mathfrak{L}$ relies on a class of possible market models. Consequently the penalty of misspecification should only take into account the values of ρ at the positions from $\mathfrak{L}$, as stated in property $(**)$.

The general representation result w.r.t. inner regular probability measures encloses conditions which imply the properties $(*), (**)$.

Theorem 1 *Let* Δ_c $(c \in]-\rho(0), \infty[)$ *gather all* P *from* $\mathcal{M}_1(\mathcal{S})$ *with* $\alpha_\rho(P) \leq c$, *and let* ρ *satisfy the following properties.*

(1) $\lim_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ *for every nonnegative* $X \in \mathfrak{X}$ *and* $\lambda > 0$,

(2) $\rho(X) = \sup_{X\leq Y\in E} \inf_{Y\geq Z\in\mathfrak{X}} \rho(Z)$ *for all nonnegative bounded* $X \in \mathfrak{X}$,

(3) $\rho(X_n) \searrow \rho(X)$ *for any isotone sequence* $(X_n)_n$ *of bounded positions* $X_n \in \mathfrak{L}$ *with* $X_n \nearrow X \in \mathfrak{L}$, *X bounded,*

(4) $\inf_{Y\geq Z\in\mathfrak{X}} \rho(Z) = \inf_{Y\geq Z\in\mathfrak{L}} \rho(Z)$ *for* $Y \in E$.

Then we may state:

.1 The initial topology $\tau_{\mathfrak{L}}$ *on* $\mathcal{M}_1(\mathcal{S})$ *induced by the mappings* $\psi_X : \mathcal{M}_1(\mathcal{S}) \to \mathbb{R}$, $\mathrm{P} \mapsto E_{\mathrm{P}}[X]$, $(X \in \mathfrak{L})$ *is completely regular and Hausdorff.*

.2 Each Δ_c $(c \in]-\rho(0), \infty[)$ *is compact w.r.t.* $\tau_{\mathfrak{L}}$, *and furthermore for every* Λ *from the domain of* β_ρ *there is some* $\mathrm{P} \in \mathcal{M}_1(\mathcal{S})$ *with* $\Lambda|\mathfrak{L} = E_{\mathrm{P}}|\mathfrak{L}$ *and* $\alpha_\rho(\mathrm{P}) \leq \beta_\rho(\Lambda)$.

.3 $\rho(X) = \sup_{\mathrm{P}\in\mathcal{M}_1(\mathcal{S})} (E_{\mathrm{P}}[-X] - \alpha_\rho(\mathrm{P}))$ *for all* $X \in \mathfrak{X}$.

Statement .1 is borrowed from [14] (p.12 there), the proof of the remaining parts of Theorem 1 may be found in [15] (section 7).

Remarks 1 *Assumption (1) is just the cutting condition as discussed in the previous section, whereas assumption (2) is the necessary regularity condition from Proposition 5. The continuity property (3) combined with the cutting condition yield property (*). In view of Theorem 2 property (*) is even equivalent with (1), (3). Finally the assumptions (1), (4) imply property (**). Moreover, the conditions* $(*), (**)$ *together are equivalent with the assumptions* $(1), (3), (4)$.

Remarks 2 *Let us point out some special situations where the assumptions on* ρ, *imposed in Theorem 1, may be simplified:*

.1 If $\mathfrak{X}$ *is restricted to bounded positions, then assumption (1) is redundant. Also (2), (4) hold in general in the case* $\mathfrak{X} = \mathfrak{L}$.

.2 Assumption (3) is fulfilled in general whenever $\mathfrak{L}_{+b}$, *consisting of all nonnegative bounded* $X \in \mathfrak{L}$, *is a so-called* **Dini cone**, *i.e.* $\inf_n \sup_{\omega\in\Omega} X_n(\omega) = \sup_{\omega\in\Omega} \inf_n X_n(\omega)$ *for any antitone sequence* $(X_n)_n$ *in* $\mathfrak{L}_{+b}$ *with pointwise limit in* $\mathfrak{L}_{+b}$. *The most prominent Dini cones are the cones of nonnegative upper semicontinuous and nonnegative continuous real-valued mappings on compact Hausdorff spaces due to the general Dini lemma (cf. [9], Theorem 3.7).*

.3 If $E \subseteq \mathfrak{X}$, *then assumptions (1), (2) read as follows:*

(1) $\rho(X) = \sup_{X\leq Y\in E} \rho(Y)$ *for all nonnegative bounded* $X \in \mathfrak{X}$,

(2) $\rho(Y) = \inf_{Y\geq Z\in\mathfrak{L}} \rho(Z)$ *for* $Y \in E$.

Let us now consider some special situations where Theorem 1 might be used.

Remark 2 *Let* $\Omega = \widetilde{\Omega} \times \mathbb{T}$ *with* $\widetilde{\Omega}$ *denoting a set of scenarios, equipped with a metrizable topology* $\tau_{\widetilde{\Omega}}$ *as well as the induced* $\sigma-$*algebra* $\mathcal{B}(\widetilde{\Omega})$, *and* $\mathbb{T}$ *being a time set, endowed with a separably metrizable topology* $\tau_{\mathbb{T}}$ *as well as the generated* $\sigma-$*algebra* $\mathcal{B}(\mathbb{T})$. *Furthermore let* $\mathfrak{X}$ *consist of all bounded real-valued mappings on* $\widetilde{\Omega} \times \mathbb{T}$ *which are measurable w.r.t. the product* $\sigma-$*algebra* $\mathcal{B}(\widetilde{\Omega})\otimes\mathcal{B}(\mathbb{T})$, *and let* $\mathfrak{L}$ *be the set of all bounded real-valued mappings on* $\widetilde{\Omega} \times \mathbb{T}$ *which are continuous w.r.t. the product topology* $\tau_{\widetilde{\Omega}} \times \tau_{\mathbb{T}}$. *Finally* $\mathcal{S}$ *is defined to gather the closed subsets of* $\widetilde{\Omega} \times \mathbb{T}$ *w.r.t. the metrizable topology* $\tau_{\widetilde{\Omega}} \times \tau_{\mathbb{T}}$. *Using the introduced notations,* $\sigma(\mathfrak{X}) = \mathcal{B}(\Omega)\otimes\mathcal{B}(\mathbb{T})$, *the product* $\sigma-$*algebra of* $\mathcal{B}(\Omega)$ *and* $\mathcal{B}(\mathbb{T})$, *is generated by* $\mathcal{S}$, $\mathfrak{L} \subseteq \mathfrak{X}$, *and we may restate Theorem 1 with E being the space of all bounded nonnegative lower semicontinuous mappings on* $\Omega \times \mathbb{T}$. *This version generalizes an analogous result for the one-period positions (cf. [13], Theorem 2)*

We may also utilize Theorem 1 for cadlag positions.

Remark 3 *Let* $\mathbb{T} = [0, T], \mathfrak{C} = \mathbb{R}$, *let* $(\mathcal{F}_t)_{t\in\mathbb{T}}$ *be a filtration of* $\sigma-$*algebras on some nonvoid set* $\widetilde{\Omega}$, *and let* $\mathfrak{X}$ *be the set of cadlag positions, i.e. mappings* $X \in \mathbb{R}^{\widetilde{\Omega}\times\mathbb{T}}$ *such that* $X(\cdot, t)$ *is* $\mathcal{F}_t-$*measurable for every* $t \in \mathbb{T}$ *and* $X(\omega, \cdot)$ *is a cadlag function for any* $\omega \in \widetilde{\Omega}$. *Then* $\sigma(\mathfrak{X})$ *is the so-called optional* $\sigma-$*algebra. We may associate for stopping times* S_1, S_2, $S_1 \leq S_2$, *the stochastic interval* $[S_1, S_2[$, *defined by* $[S_1, S_2[(\omega, t) := 1$ *if* $S_1(\omega) \leq t < S_2(\omega)$, *and* $[S_1, S_2[(\omega, t) := 0$ *otherwise.* $\mathfrak{I}$ *stands for the set of all such stochastic intervals. It can be shown that* $\sigma(\mathfrak{X})$ *is generated by the stochastic intervals* $[S, \infty[$ *(cf. [3], IV, 64).*

For $\mathfrak{L}$ *let us choose the vector space spanned by the stochastic intervals* $[S, \infty[$. *Using the introduced notations, we may restate Theorem 1.*

Remark 4 *Recently, convex risk measures has been used as objectives of optimization problems like e.g. the investment for asset allocations or the choice of consumption-investment plans, when the investor is risk- and ambiguity-averse (cf. e.g. [19], [22]). Then Theorem 1 provides not only a criterion which recognizes an investor with such an risk attitude, but it* *might be also the starting point to get on to tracks of robust expected utility maximization. In particular the compactness statement .2 of Theorem 1 may allow to employ duality methods for the optimization problems.*

4 Strong $\sigma-$additive robust representation of convex risk measures

We want to look for conditions which induce a strong robust representation of ρ by probability measures in the sense that

$$\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1}(-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$$

holds for any $X \in \mathfrak{X}$. The considerations are reduced to a Stonean vector lattice $\mathfrak{X}$ being stable w.r.t. countable convex combinations of antitone sequences of financial positions. In this case the following result gives a complete answer to the problem of strong robust representations.

Theorem 2 *Let $\mathfrak{X}$ be a Stonean vector lattice and let us assume that for every antitone sequence $(X_n)_n$ in $\mathfrak{X}$ with $X_n \searrow 0$ and each sequence $(\lambda_n)_n$ in $[0,1]$ with $\sum_{n=1}^{\infty}\lambda_n = 1$ there is some pointwise limit $\sum_{n=1}^{\infty}\lambda_n X_n$ of $(\sum_{n=1}^{m}\lambda_n X_n)_m$ belonging to $\mathfrak{X}$. Then the following statements are equivalent:*

.1 $\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1}(-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ holds for every $X \in \mathfrak{X}$.

.2 $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$.

.3 Λ is representable by a probability measure from $\mathcal{M}_1$ for $\beta_\rho(\Lambda) < \infty$.

The implication .2 ⇒ .3 may be concluded from Theorem 1, whereas .3 ⇒ .1 is trivial due to Proposition 1. The proof of the implication .1 ⇒ .2 may be found in [15] (section 9), its crucial tool is Simons' lemma (cf. [20], Lemma 2). For application of this result we need the assumed stability w.r.t. countable convex combinations of positions.

Remark 5 *The continuity property .2 in Theorem 2 is implied by a technically simplier one, which is even equivalent in many cases (cf. [15], Theorem 4.1).*

For bounded one-period positions, Theorem 2 enables us to give an equivalent characterization of convex risk measures that admit strong robust representations by probability measures.

Corollary 1 *Let $\mathcal{F}$ denote some $\sigma-$algebra on Ω, and let $\mathfrak{X}$ consist of all bounded $\mathcal{F}-$measurable real-valued mappings. Then the following statements are equivalent:*

.1 $\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1}(-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ holds for every $X \in \mathfrak{X}$

.2 $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$.

Originally, the implication .2 ⇒ .1 of Corollary 1 may be found in [5], whereas the full equivalence has been shown the first time in [13].

In the case of $\mathfrak{X} = \mathcal{L}_\infty(\Omega,\mathcal{F},\mathrm{P})$, the set of all essentially bounded mappings w.r.t. a reference probability measure P on a $\sigma-$algebra $\mathcal{F}$, we may retain immediately the equivalent characterization of strong robust representations for ρ shown in [8], where the identity on $\mathbb{R}$ has been chosen for the price functional π. Note that the condition $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$ P −a.s. is equivalent with the property $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$.

Corollary 2 *Let $\mathfrak{X} = \mathcal{L}_\infty(\Omega,\mathcal{F},\mathrm{P})$, and let ρ satisfy $\rho(X) = \rho(Y)$ for $X = Y$ P a.s.. Then the representation $\rho(X) = \max_{\mathrm{Q}\in\mathcal{M}_1}(-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q}))$ holds for all $X \in \mathcal{L}_\infty(\Omega,\mathcal{F},\mathrm{P})$ if and only if $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$ P −a.s..*

Remark 6 *Besides the potential for robust expected utility maximization as emphasized in Remark 4, Theorem 2 has significance from the practical point of view. In many cases the calculation of outcomes of risk measures has to be employed by numerical optimization algorithms, and the most customary ones assume the existence of solutions. Therefore Theorem 2 can be used to check whether the desired algorithms may be applied.*

5 Representation of convex risk measures by probability measures and the Fatou properties

In Proposition 4 we have indicated the Fatou property and continuity from above as necessary conditions for a $\sigma-$additive robust representation of the convex risk measure ρ. They are even sufficient if a market model is available for the investor, choosing the identity on $\mathbb{R}$ for the price functional (cf. [5]). As pointed by Delbaen (in [2]), they are not sufficient in general for a robust representation of ρ by ($\sigma-$additive) probability measures, even if $\mathfrak{X}$ contains bounded positions only. It will turn out by the investigations within this section that in the case of uncertainty about the market model the nonsequential counterpart of the Fatou property takes over partly the role that the Fatou property plays when a reference probability measure is given. We shall say that ρ satisfies the **nonsequential Fatou property** if $\liminf_i \rho(X_i) \geq \rho(X)$ holds whenever $(X_i)_{i\in I}$ is a uniformly bounded net in $\mathfrak{X}$ which converges pointwise to some bounded $X \in \mathfrak{X}$.

At least for the sufficiency of the Fatou property we need further assumptions on the space $\mathfrak{X}$ of available positions. Since both Fatou properties are related to the pointwise topology on the space $B(\Omega)$, gathering the bounded real-valued mappings on Ω, we shall impose additional assumptions on this topology. The idea is to modify in view of Proposition 1 the classical proofs for the case of a reference probability measure, using again the Krein-Smulian theorem. Justified by success we shall use the following conditions.

(5.1) For any $r > 0$, every $Z \in \mathfrak{X}_b$ from the closure of $A_r := \{X \in \mathfrak{X}_b \mid \rho(X) \leq 0, \sup_{\omega\in\Omega} |X(\omega)| \leq r\}$ w.r.t. the topology of pointwise convergence on $\mathfrak{X}_b$ is the pointwise limit of a sequence in A_r.

(5.2) The sets $B_r := \{X \in \mathfrak{X}_b \mid \sup_{\omega\in\Omega} |X(\omega)| \leq r\}$ $(r > 0)$ are closed w.r.t. the topology of pointwise convergence on $B(\Omega)$.

Assumption (5.1) provides an important special situation when the Fatou property and its nonsequential counterpart are equivalent.

Lemma 1 *Under (5.1) ρ satisfies the nonsequential Fatou property if and only if it fulfills the Fatou property.*

The proof is enclosed in section 9 of [15].

Remark 7 *The sequential condition (5.1) is closely related with the concepts of double limit relations. For a comprehensive exposition the reader is referred to [12]. In general one may try to apply double limit relations to $\mathfrak{X}_b$ and suitable sets of bounded countably additive set functions on $\sigma(\mathfrak{X})$.*

We are now ready for the main result of this section.

Theorem 3 *Let either $\mathfrak{X} = \mathfrak{X}_b$ or $\mathfrak{X}$ be a Stonean vector lattice such that $\lim_{n\to\infty} \rho(\lambda(X - n)^+) = \rho(0)$ holds for any nonnegative $X \in \mathfrak{X}$, $\lambda > 0$. Furthermore let $\alpha^{-1}(\mathbb{R}) \neq \emptyset$. Consider the following statements:*

.1 ρ satisfies the nonsequential Fatou property.

.2 ρ has a $\sigma-$additive robust representation w.r.t. $\mathcal{M}_1$.

.3 ρ fulfills the Fatou property.

If (5.2) is valid, then .1 $\Rightarrow$ .2 $\Rightarrow$ .3, and all statements are equivalent provided that condition (5.1) holds in addition.

The proof may be found in section 9 of [15].

Remark 8 *The nonsequential Fatou property is not necessary for a $\sigma-$additive representation of risk measures. Take for example $\mathfrak{X}$ the space of all boundend Borel-measurable mappings on $\mathbb{R}$, and define ρ by $\rho(X) = -E_{\mathrm{P}}[X]$, where P denotes any probabality measure which is absolutely convex w.r.t. the Lebesgue-Borel measure on $\mathbb{R}$. Obviously, on one hand ρ is a convex risk measure w.r.t. the identity on $\mathbb{R}$, having a trivial $\sigma-$additive robust representation. On the other hand, consider the net $(X_i)_{i\in I}$ of all indicator mappings of the cofinite subsets of $\mathbb{R}$, directed by set inclusion. It converges pointwise to 0, but unfortunately $\liminf_i \rho(X_i) = -1 < 0 = \rho(0)$.*

In the case of an at most countable Ω, we have a simplified situation which admits an application of the full Theorem 3. The reason is that then the topology of pointwise convergence on the space $B(\Omega)$ is metrizable.

Corollary 3 *Let Ω be at most countable, and let $\mathfrak{X} \subseteq B(\Omega)$ be sequentially closed w.r.t. the pointwise topology on $B(\Omega)$. Then ρ has a robust representation by probability measures from $\mathcal{M}_1$ if and only it satisfies the Fatou property, or equivalently, if and only if ρ is continuous from above.*

Remark 9 *Let a market model with reference probability measure P be given, and let $\mathfrak{X} := \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ be the space of all $\mathrm{P}-$essentially bounded mappings on Ω. Furthermore ρ is supposed to be a convex risk measure w.r.t. the identity on $\mathbb{R}$, satisfying $\rho(X) = \rho(Y)$ for $X = Y$ $\mathrm{P}-a.s.$. We may apply the full Theorem 3 (cf. section 9 in [15]) to retain an equivalent characterization of the robust representations for ρ which may be found in [5] (Theorem 4.31). More precisely, if $\mathcal{M}_1(\mathrm{P})$ denotes the set of probability measures on $\mathcal{F}$ which are absolutely continuous w.r.t. P, then the following statements are equivalent.*

.1 $\rho(X) = \sup_{\mathrm{Q}\in\mathcal{M}_1(\mathrm{P})} (-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q}))$ for all X from $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$.

.2 $\rho(X_n) \nearrow \rho(X)$ for $X_n \searrow X$ $\mathrm{P}-a.s.$.

.3 $\liminf_{n\to\infty} \rho(X_n) \geq \rho(X)$ whenever $(X_n)_n$ is a uniformly $\mathrm{P}-$essentially bounded sequence in $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ with $X_n \to X$ $\mathrm{P}-a.s.$.

It is unclear whether we may avoid in Theorem 3 condition (2.2) in order to guarantee a $\sigma-$additive robust representation of convex risk measures by the nonsequential Fatou property. Moreover, the nonsequential Fatou property is unsatisfactory in the way that it does not work for trivial representations like those indicated in Remark 8. However, we may only guarantee a sufficient substitution by the Fatou property under the quite restrictive condition (2.1). So it seems that in presence of model uncertainty the Fatou property and its nonsequential counterpart are appropriate conditions for $\sigma-$additive representations of convex risk measures in quite exceptional situations only, like an at most countable Ω.

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