Measuring Uncertainty with Imprecision Indices

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Abstract

The paper is devoted to the investigation of imprecision indices, introduced in [8]. They are used for evaluating uncertainty (namely imprecision), which is contained in information, described by fuzzy (non-additive) measures, in particular, by lower and upper probabilities. We argue that there exist various types of uncertainty, for example, randomness, investigated in probability theory, imprecision, described by interval calculi, inconsistency, incompleteness, fuzziness and so on. In general these types of uncertainty have very complex behavior, caused by their interaction. Therefore, the choice of uncertainty measures is not unique, and depends on the problems addressed. The classical uncertainty measures are Shannon's entropy and Hartley's measure. In the paper imprecision indices and also linear ones are introduced axiomatically. The system of axioms allows us to define various imprecision indices. So we investigate the algebraic structure of all imprecision indices and investigate their families with best properties.

Keywords. Imprecision indices, lower and upper probabilities, uncertainty-based information.

1 Introduction

Measuring uncertainty plays a major role in uncertainty theories, in particular, probability theory, information theory, fuzzy sets theory and so on. There are some ways how to define such measures in the theory of evidence, in the theory of fuzzy (non-additive) measures and in the theory of imprecise probabilities. However, one can see that in such general theories the uncertainty measure with the best properties has not been found as yet. This situation is explained by the very complex interaction among various types of uncertainty, including randomness, inconsistence, imprecision, incompleteness of the analyzed information. We recall classical uncertainty measures, used in information theory and probability theory. Let X be a finite set of alternatives. Assigning to each alternative $x \in X$ some probability $P(\{x\})$, we have information, which is described by probability measure P, and in this case Shannon's entropy $S(P) = -\sum_{x\in X} P(\{x\})\log_2 P(\{x\})$ can be used. Let we know only that the "true" alternative is in a nonempty set $B \subseteq X$. This situation can be described by the non-additive measure $\eta_{\langle B\rangle}(A) = \begin{cases} 1, B \subseteq A \\ 0, B \subsetneq A \end{cases}$, $A \subseteq X$, which gives the lower probability of an event A, and Hartley's measure $H(\eta_{\langle B\rangle}) = \log_2|B|$ can be justified. It is easily seen that in the first case uncertainty has a type that one call randomness, and the second case is more connected with imprecision of the information. The generalization of these two cases consists in the following. Consider a pair $(\underline{g}, \overline{g})$ of set functions $\underline{g}: 2^X \to [0,1]$, $\overline{g}: 2^X \to [0,1]$ defined on the powerset 2^X. We suggest that $\underline{g}(A) \le \overline{g}(A)$ for all $A \in 2^X$, $\underline{g}(\varnothing) = \overline{g}(\varnothing) = 0$, and there is a "true" probability measure P on 2^X with $\underline{g}(A) \le P(A) \le \overline{g}(A)$ for all $A \in 2^X$. In other words, set functions $\underline{g}, \overline{g}$ give us upper and lower bounds of probabilities, and for any event $A \in 2^X$ we have only the interval $[\underline{g}(A), \overline{g}(A)]$ of possible values of a "true" probability $P(A)$. In practical issues it is sufficient to define the lower probability $\underline{g}$, the upper probability can be calculated by $\overline{g}(A) = 1 - \underline{g}(\overline{A})$, where $A \in 2^X$ and $\overline{A}$ is the complement of A. Due to works of Abellan, Klir, Higashi, Harmanec and others (see [1,5,6,7]), there are two important uncertainty measures, which show the best properties in a sense of obeying axioms, which are similar to the axioms of Shannon's entropy. They are generalized Hartley's measure, and aggregate measure of uncertainty. Let $\underline{g}$ be a belief function, i.e. it can be represented by $\underline{g}=\sum_{B\in 2^X} m(B)\eta_{\langle B\rangle}$, where $m(\emptyset)=0$, $m(B)\geq 0$ for all $B\in 2^X$, and $\sum_{B\in 2^X} m(B)=1$. Then generalized Hartley's measure is defined by

$$GH\left(\underline{g}\right)=\sum_{B\in 2^X\setminus\{\emptyset\}} m(B)\log_2|B| .$$

The aggregate measure of uncertainty is calculated by

$$Au\left(\underline{g}\right)=\sup_{P\geq \underline{g}} S(P) ,$$

where sup is taken over all probability measures on 2^X, which are consistent with $\underline{g}$, i.e. $P(A)\geq \underline{g}(A)$ for all $A\in 2^X$. It is worth to mention that generalized Hartley's measure can be used for measuring imprecision and aggregate measure of uncertainty for total uncertainty. It is easy to check that aggregate measure of uncertainty coincides with Shannon's entropy for probability measures and with Hartley's measure for $\underline{g}=\eta_{\langle B\rangle}$, $B\neq\emptyset$.

The paper has the following structure. We remind first some definitions and results from the theory of non-additive measures and axiomatic of imprecision indices, formulated in [8]. Then we analyze so called linear imprecision indices on the set of upper and lower probabilities, giving their detailed description, and introducing their important families with symmetrical properties. We finish the paper with generalizing imprecision indices for the set of all monotone measures introducing in addition indices of inconsistency.

2 Basic definitions and problem statement

Let X be a finite set. In the sequel we will use the following notations:

1. M is the set of all real-valued set functions on the powerset 2^X;

2. $M_0=\left\{g\in M \mid g(\emptyset)=0\right\}$;

3. We write $g_1\leq g_2$ for $g_1,g_2\in M$ if $g_1(A)\leq g_2(A)$ for all $A\in 2^X$.

4. $M_{mon}\subset M_0$ is the set of all normalized monotone set functions on 2^X. It means that $g\in M_{mon}$ implies $g(\emptyset)=0$, $g(X)=1$, and $g(A)\leq g(B)$ if $A\subseteq B$.

5. $M_{\Pr}$ is the set of all probability measures on 2^X;

6. $M_{low}=\left\{g\in M_0 \mid \exists P\in M_{\Pr}: g\leq P\right\}$ is the set of all lower probabilities on 2^X.

6. $M_{up}=\left\{g\in M_0 \mid \exists P\in M_{\Pr}: g\geq P\right\}$ is the set of all upper probabilities on 2^X.

7. Let $g\in M$ then the dual of g is denoted by $\overline{g}$ and by definition: $\overline{g}(A)=g(X)-g\left(\overline{A}\right)$, $A\in 2^X$.

8. M_{bel} is the set of all belief functions on 2^X. Any $g\in M_{bel}$ has the following unique representation: $g=\sum_{B\in 2^X} m(B)\eta_{\langle B\rangle}$, where $m(B)\geq 0$ for all $B\in 2^X$, $m(\emptyset)=0$, and $\sum_{B\in 2^X} m(B)=1$.

9. M_{pl} is the set of all plausibility functions on 2^X. Any $g\in M_{pl}$ is represented uniquely by $g=\sum_{B\in 2^X} m(B)\overline{\eta}_{\langle B\rangle}$, where $m(B)\geq 0$ for all $B\in 2^X$, $m(\emptyset)=0$, and $\sum_{B\in 2^X} m(B)=1$.

We can consider the set M (or M_0) as a linear space w.r.t. to usual sum of set functions and usual product of set functions and real numbers. In non-additive measure theory, the basis, consisting of functions $\eta_{\langle B\rangle}$, $B\in 2^X$, is of interest. Let $g\in M$ and $g=\sum_{B\in 2^X} m_g(B)\eta_{\langle B\rangle}$ then the set function m_g is called Möbius transform of g. The function m_g is expressed by $m_g(B)=\sum_{A:A\subseteq B}(-1)^{|B\setminus A|}g(A)$. We will also use so-called dual Möbius transform of g. This transform is connected with the basis, consisting of set functions $\eta^{\langle B\rangle}$, $B\in 2^X$, defined by $\eta^{\langle B\rangle}(A)=\eta_{\langle\overline{B}\rangle}(\overline{A})$. Let $g=\sum_{B\in 2^X} m^g(B)\eta^{\langle B\rangle}$ then the set function m^g is called dual Möbius transform of g. It is calculated by $m^g(B)=\sum_{A:B\subseteq A}(-1)^{|A\setminus B|}g(A)$.

We remind now some definitions, introduced in [8].

Definition 1. A functional $f:M_{low}\to[0,1]$ is called *imprecision index* if the following conditions are fulfilled: 1) $g\in M_{\Pr}$ implies $f(g)=0$; 2) $f(g_1)\geq f(g_2)$ for all $g_1,g_2\in M_{low}$ such that $g_1\leq g_2$; 3) $f\left(\eta_{\langle X\rangle}\right)=1$.

Remark 1. We write $g_1<g_2$ for $g_1,g_2\in M$ if $g_1\leq g_2$ and $g_1\neq g_2$. Then *sensitive imprecision indices* have to obey: $f(g_1)>f(g_2)$ if $g_1,g_2\in M_{low}$ and $g_1<g_2$. In some works (e.g. [5,7]) there is an argumentation that uncertainty measures have to obey also subaddivity

property. Here we do not discuss this problem, because, in our opinion, this property is related to another kind of uncertainty, which can be called incompleteness of the information. However, adding the subadditivity property to the list of axioms for imprecision indices on M_{low} leads to the fact that there is no sensitive imprecision index with subadditivity property (for checking this statement you can use Example 1 in [1]).It is clear that there are many ways for defining imprecision indices. One class of them consisting of linear imprecision indices is described in the following definition.

Definition 2. An imprecision index f on M_{low} is called *linear* if for any linear combination $\sum_{j=1}^{k}\alpha_j g_j \in M_{low}$, $\alpha_j \in \mathbb{R}$, $g_j \in M_{low}$, $j=1,...,k$, we have $f\left(\sum_{j=1}^{k}\alpha_j g_j\right)= \sum_{j=1}^{k}\alpha_j f\left(g_j\right)$.

3 The investigation of linear imprecision indices

We notice first that any linear functional f on M is defined uniquely by its values on a chosen basis of M. This enables to define f by the set function $\mu_f : 2^X \to \mathbb{R}$ with the following property $\mu_f(B) = f\left(\eta_{\langle B\rangle}\right)$, $B \in 2^X$. Since any $g \in M_{low}$ is represented as a linear combination of $\left\{\eta_{\langle B\rangle}\right\}_{B\in 2^X\setminus\{\varnothing\}}$, we take by definition that $\mu_f(\varnothing)=0$ (or $f\left(\eta_{\langle\varnothing\rangle}\right)=0$) for any linear imprecision index f.

Proposition 1 [8]. *Let f be a linear imprecision index on M_{low} then $\mu_f \in M_{mon}$ with $\mu_f\left(\{x\}\right)=0$ for any $x \in X$.*

The following proposition gives us the expression of any linear functional through the values of the transformed set function.

Proposition 2. *Let f be a linear functional on M then $f(g)=\sum_{B\in 2^X} m^{\mu_f}(B)g(B)$ for any $g \in M$.*

Proof. By definition $\mu_f = \sum_{B\in 2^X} m^{\mu_f}(B)\eta^{\langle B\rangle}$ and $g = \sum_{C\in 2^X} m_g(C)\eta_{\langle C\rangle}$, therefore,

$$f(g)=\sum_{C\in 2^X} m_g(C)\mu_f(C)$$
$$=\sum_{C\in 2^X} m_g(C)\sum_{B\in 2^X} m^{\mu_f}(B)\eta^{\langle B\rangle}(C)$$
$$=\sum_{B\in 2^X} m^{\mu_f}(B)\sum_{C\in 2^X} m_g(C)\eta_{\langle C\rangle}(B)$$
$$=\sum_{B\in 2^X} m^{\mu_f}(B)g(B). \blacksquare$$

The following theorem gives necessary and sufficient conditions on a linear functional to be an imprecision index through the dual Möbius transform of μ_f.

Theorem 1. *Let f be a linear functional on M then it is an imprecision index on M_{low} iff*

a) $m^{\mu_f}(X)=1$; $\sum_{D\in 2^X} m^{\mu_f}(D)=0$;

b) $\sum_{D:\, x\in D} m^{\mu_f}(D)=0$ *for all* $x\in X$;

c) $m^{\mu_f}(D)\le 0$ *for all* $D\in 2^X\setminus\{\varnothing, X\}$.

Proof. It is clear that the condition a) guarantees that $f\left(\eta_{\langle X\rangle}\right)=1$ and $f\left(\eta_{\langle\varnothing\rangle}\right)=0$. It is easy to show that b) is the necessary and sufficient condition that $f(g)=0$ for any $g\in M_{\Pr}$. Indeed, since $\eta_{\langle\{x\}\rangle}\in M_{\Pr}$ then $\mu_f\left(\{x\}\right)=f\left(\eta_{\langle\{x\}\rangle}\right)=0$, $\mu_f\left(\{x\}\right)=\sum_{D:\, x\in D} m^{\mu_f}(D)=0$. On the other hand, any $g\in M_{\Pr}$ can be represented as a convex sum of $\eta_{\langle\{x\}\rangle}$, i.e. $g=\sum_{x\in X} m_g\left(\{x\}\right)\eta_{\langle\{x\}\rangle}$, hence,

$$f(g)=\sum_{x\in X} m_g\left(\{x\}\right)f\left(\eta_{\langle\{x\}\rangle}\right)$$
$$=\sum_{x\in X} m_g\left(\{x\}\right)\mu_f\left(\{x\}\right)=0.$$

So b) is proved. c) is the sufficient and necessary condition of antimonotonicity of f on M_{low}. Let c) be fulfilled and $g_1\le g_2$ for $g_1, g_2\in M_{low}$ then by Proposition 2

$$f(g_1)-f(g_2) = \sum_{B\in 2^X} m^{\mu_f}(B)\left(g_1(B)-g_2(B)\right)$$
$$= \sum_{B\in 2^X\setminus\{\varnothing,X\}} m^{\mu_f}(B)\left(g_1(B)-g_2(B)\right).$$

Since $g_1(B)-g_2(B)\le 0$ for any $B\in 2^X$ and $m^{\mu_f}(B)\le 0$ for any $B\in 2^X\setminus\{\varnothing,X\}$, we get $f(g_1)\ge f(g_2)$, i.e. c) implies antimonotonicity of f. Vice versa, let f be antimonotone on M_{low} then for any $D\in 2^X\setminus\{\varnothing,X\}$ we can always find such $g_1, g_2\in M_{low}$ with $g_1(B)=g_2(B)$ for all $B\ne D$, and $g_1(D)<g_2(D)$. According to Proposition 2 $0\le f(g_1)-f(g_2)= m^{\mu_f}(D)(g_1(D)-g_2(D))$, i.e. $m^{\mu_f}(D)\le 0$. ■

Conditions of Theorem 1 can be transformed to the form, which is very close to the condition "avoiding sure loss" from the theory of imprecise probabilities [10]. It enables to get the implicit expression for an arbitrary linear imprecision index. We will further use the functions 1_B, $B\subseteq X$, on X defined by $1_B(x)=1$ if $x\in B$, and $1_B(x)=0$ otherwise.

Theorem 2. *Any linear imprecision index* f *on* M_{low} *can be uniquely represented by*

$$f(g)=1-\sum_{B\in 2^X} m(B)g(B),$$

where the set function m *obeys the following conditions:*
1) $m(\varnothing)=0$, $m(X)=0$, $m(B)\geq 0$ *for all* $B\in 2^X$;
2) $\sum_{B\in 2^X} m(B)1_B=1_X$.

Remark 2. The condition of "avoiding sure loss" from the theory of imprecise probabilities can be formulated with the help of the set function m from the Theorem 2 as follows: let $g\in M_0$ then $g\in M_{low}$ iff for any set function m obeying 1), 2) from Theorem 2, we have $\sum_{B\in 2^X} m(B)g(B)\leq 1$.

Theorem 3. *Let* f *be a linear functional on* M *then it is an imprecision index on* M_{low} *iff* $\mu_f=a\mu-b\overline{\eta}_{\langle X\rangle}$, *where* $b>0$, $a=1+b$, *and* $\mu\in M_{pl}$ *with* $\mu(\{x\})=b/a$ *for all* $x\in X$.

Proof. *Necessity.* Let f be a linear imprecision index on M_{low} then

$$\mu_f(B)=\sum_{A\in 2^X\setminus\{X,\varnothing\}} m^{\mu_f}(A)\eta_{\langle\overline{A}\rangle}(\overline{B})$$
$$+\,m^{\mu_f}(X)\eta_{\langle\varnothing\rangle}(\overline{B})+m^{\mu_f}(\varnothing)\eta_{\langle X\rangle}(\overline{B}),$$

where $m^{\mu_f}(A)\leq 0$ for any $A\in 2^X\setminus\{X,\varnothing\}$ and $\eta_{\langle\varnothing\rangle}\equiv 1$, $m^{\mu_f}(X)=1$. Let $a=-\sum_{A\in 2^X\setminus\{X,\varnothing\}} m^{\mu_f}(A)$ then taking $q(A)=-\frac{1}{a}m^{\mu_f}(\overline{A})$ for $A\in 2^X\setminus\{X,\varnothing\}$ and $m(A)=0$ for $A\in\{X,\varnothing\}$, we get

$$\mu_f(B)=-a\sum_{A\in 2^X} q(\overline{A})\eta_{\langle\overline{A}\rangle}(\overline{B})+1+m^{\mu_f}(\varnothing)\eta_{\langle X\rangle}(\overline{B})$$
$$=a\sum_{A\in 2^X} q(A)\left(1-\eta_{\langle A\rangle}(\overline{B})\right)$$
$$-\,m^{\mu_f}(\varnothing)\left(1-\eta_{\langle X\rangle}(\overline{B})\right)+m^{\mu_f}(\varnothing)+1-a\,.$$

It is clear $m^{\mu_f}(\varnothing)+1-a=\sum_{A\in 2^X} m^{\mu_f}(A)=\mu_f(\varnothing)=0$, hence, we get the representation required

$$\mu_f(B)=a\sum_{A\in 2^X} q(A)\overline{\eta}_{\langle A\rangle}(B)-b\overline{\eta}_{\langle X\rangle}(B),$$

where $\mu=\sum_{A\in 2^X} q(A)\overline{\eta}_{\langle A\rangle}$, $b=m^{\mu_f}(\varnothing)$, $a=1+b$.

It is easy to show that $\mu(\{x\})=b/a$, $x\in X$, and $b>0$. Actually, by Proposition 1 $\mu_f(\{x\})=0$ for all $x\in X$, i.e. $\mu(\{x\})=b/a$ for all $x\in X$. On the other hand,

$$\mu_f(\{x\})=a\sum_{A:x\in A} q(A)-b=0,$$

i.e. $b\geq 0$ and if $b=0$ then $q\equiv 0$ and this contradicts to the definition of imprecision index.

Sufficiency. Assume that we have the representation of μ_f from the theorem. We prove sufficiency if we check all conditions from Theorem 1. We see that $\mu_f(\varnothing)=0$, $\mu_f(X)=1$, and $\mu_f(\{x\})=0$ for all $x\in X$, i.e. conditions a), b) are true. We will further prove that $m^{\mu_f}(A)\leq 0$ for all $A\in 2^X\setminus\{\varnothing,X\}$. Since μ is a plausibility function, it is represented by $\mu=\sum_{A\in 2^X} q(A)\overline{\eta}_{\langle A\rangle}$, where $q(A)\geq 0$ for all $A\in 2^X$, $q(\varnothing)=0$, and $\sum_{A\in 2^X} q(A)=1$. We can write

$$\mu_f(B)=a\sum_{A\in 2^X} q(A)\overline{\eta}_{\langle A\rangle}(B)-b\overline{\eta}_{\langle X\rangle}(B)$$
$$=a\sum_{A\in 2^X} q(A)\left(1-\eta_{\langle A\rangle}(\overline{B})\right)-b\left(1-\eta_{\langle X\rangle}(\overline{B})\right)$$
$$=a\sum_{A\in 2^X} q(\overline{A})\left(1-\eta_{\langle\overline{A}\rangle}(\overline{B})\right)-b\left(1-\eta_{\langle X\rangle}(\overline{B})\right).$$

The last expression implies $m^{\mu_f}(A)=-aq(\overline{A})\leq 0$ for all $A\in 2^X\setminus\{\varnothing,X\}$, i.e. c) is also true. ■

From the proof of Theorem 3, we see that we can use the basis $\left\{\overline{\eta}_{\langle B\rangle}\right\}_{B\in 2^X\setminus\{\varnothing\}}$ of M_0 for defining other sufficient and necessary conditions on linear imprecision index. We formulate them in

Corollary 1. *Let* f *be a linear functional on* M *and* $\mu_f=\sum_{A\in 2^X\setminus\{\varnothing\}} m(A)\overline{\eta}_{\langle A\rangle}$ *then* f *is an imprecision index iff 1)* $\mu_f\in M_0(X)$; *2)* $\mu_f(\{x\})=0$ *for all* $x\in X$; *3)* $m(A)\geq 0$ *for all* $A\in 2^X\setminus\{\varnothing,X\}$.

The next theorem follows from Theorem 3.

Theorem 4. *Let* f *be a linear functional on* M *then it is an imprecision index on* M_{low} *iff 1)* $\mu_f\in M_0$; *2)* $\mu_f(\{x\})=0$ *for all* $x\in X$; *3) the set function* $\mu_f^{\{x\}}$, *defined by* $\mu_f^{\{x\}}(B)=\mu_f(B\cup\{x\})$, $B\in 2^X$, *is in* M_{pl} *for any* $x\in X$.

It seems to be logical in some problems that the quantity of imprecision in the situation, where we know only that the true alternative belongs to the set B, depends on $|B|$ and does not depend on other factors. In this case we assume that $f\left(\eta_{\langle B\rangle}\right)=f\left(\eta_{\langle C\rangle}\right)$ or $\mu_f(B)=\mu_f(C)$ if $|B|=|C|$, and we call such linear imprecision indices symmetrical. In the sequel we will use the fact that such symmetrical monotone set functions can be viewed as distorted probabilities [3]. Let P be a probability measure on $X=\{x_1,...,x_N\}$; let $\lambda:[0,1]\to[0,1]$ be non-decreasing function with $\lambda(0)=0$, $\lambda(1)=1$, then the set function $g=\lambda\circ P$ ($g(A)=\lambda\left(P(A)\right)$, $A\in 2^X$) is called distorted probability. We are interested in the case, where $P(\{x_i\}) = 1/N$, $i = 1,...,N$. Further we will use the following sufficient condition of total monotonicity [2]: let $g = \lambda \circ P$, then it is a belief function if λ is infinitely differentiable on $[0,1)$ and $d^n\lambda(t)/dt^n \geq 0$, $n = 1,2,...$, for any $t \in [0,1)$.

Theorem 5. *Let f be a linear functional on M and $\mu_f = \lambda \circ P$, i.e. μ_f is a distorted probability, mentioned above, and $P(\{x_i\}) = 1/N$, $i = 1,...,N$. Then f is an imprecision index if:* 1) $\lambda(1/N) = 0$; 2) *λ is infinitely differentiable on $\left[\frac{1}{N},1\right)$ and $(-1)^{n-1} d^n\lambda(t)/dt^n \geq 0$, $n = 1,2,...$, for any $t \in \left[\frac{1}{N},1\right)$.*

Proof. We will check that the all conditions from Theorem 4 are true. It is clear that $\mu_f \in M_0$ and $\mu_f(\{x\}) = 0$ for all $x \in X$. Now we prove that 3) is also true. In this case $\mu_f^{\{x\}}(B) = \lambda(P(B \cup \{x\}))$, $B \in 2^{X\setminus\{x\}}$, $\mu_f^{\{x\}}$ can be considered as a distorted probability on $2^{X\setminus\{x\}}$, and $\mu_f^{\{x\}} = \lambda_1 \circ P_1$, where $\lambda_1(t) = \lambda\left(\frac{1+t(N-1)}{N}\right)$, $t \in [0,1]$, and $P_1(\{y\}) = 1/(N-1)$, $y \in X \setminus \{x\}$. We find that $\overline{\mu_f^{\{x\}}}(A) = 1 - \lambda_1(P_1(\bar{A})) = 1 - \lambda_1(1 - P_1(A))$, i.e. $\overline{\mu_f^{\{x\}}} = \lambda_2 \circ P_1$ is a distorted probability and $\lambda_2(t) = 1 - \lambda_1(1-t) = 1 - \lambda\left(1 - \frac{t(N-1)}{N}\right)$. It is clear $\mu_f^{\{x\}} \in M_{pl}$ iff $\overline{\mu_f^{\{x\}}} \in M_{bel}$. Then we argue that $\mu_f^{\{x\}}$ is a plausibility function if $d^n\lambda_2(t)/dt^n \geq 0$, $n = 1,2,...$, for any $t \in [0,1)$, or $(-1)^{n-1} d^n\lambda(t)/dt^n \geq 0$, $n = 1,2,...$, for any $t \in \left[\frac{1}{N},1\right)$.■

In some cases it is suitable to define symmetrical μ_f by a non-decreasing function $\varphi : [1,+\infty) \to [0,+\infty)$ with $\varphi(1) = 0$ assuming that $\mu_f(A) = \varphi(|A|)/\varphi(|X|)$ for $A \neq \emptyset$. Then $\lambda(t) = \varphi(tN)/\varphi(N)$ for $t \in \left[\frac{1}{N},1\right]$, where $N = |X|$. It is easy to see that according to Theorem 5, μ_f determines a linear imprecision index if φ is infinitely differentiable on $[1,N)$ and $(-1)^{n-1} d^n\varphi(t)/dt^n \geq 0$, $n = 1,2,...$, for any $t \in [1,N)$.

Example 1. Let $\varphi(t) = \ln(t)$ then $\mu_f(A) = \ln(|A|)/\ln(|X|)$. In this case the corresponding linear imprecision index can be considered as the analog of generalized Hartley's measure. We see that $(-1)^{n-1} d^n \ln(t)/dt^n = (n-1)!t^{-n} \geq 0$ for $t \geq 1$, i.e. μ_f determines a linear imprecision index on M_{low}.

Example 2. Given two source of information about the object of our interest. These sources are described by lower probabilities g_1 and g_2. Assume that the pointed sources are consistent, i.e. $\max\{g_1, g_2\} \in M_{low}$. We are going to use one of the sources in further analysis. This situation may be caused that we work, for example, with necessity functions and the choice of more exact information $\max\{g_1, g_2\}$ pushes out from possibility theory. Assume that we make a choice from $\{g_1, g_2\}$ using the metric on M_{mon} defined by

$$d(g_1, g_2) = \sum\nolimits_{B \in 2^X} m(B)|g_1(B) - g_2(B)|,$$

where $g_1, g_2 \in M_{mon}$, m is a weight function with $m(\emptyset) = m(X) = 0$, $m(B) > 0$ for all $B \in 2^X \setminus \{\emptyset, X\}$. We choose g_1 if $d(g_1, \max\{g_1, g_2\}) < d(g_2, \max\{g_1, g_2\})$, g_2 if $d(g_1, \max\{g_1, g_2\}) > d(g_2, \max\{g_1, g_2\})$, and if $d(g_1, \max\{g_1, g_2\}) = d(g_2, \max\{g_1, g_2\})$ then the additional analysis is needed for making a decision. Now we show how this metric is related to the notion of imprecision index. Simplifying the expression

$$\begin{gathered} d(g_1, \max\{g_1, g_2\}) - d(g_2, \max\{g_1, g_2\}) = \\ \sum_{B: g_2(B) > g_1(B)} m(B)(g_2(B) - g_1(B)) - \\ \sum_{B: g_1(B) > g_2(B)} m(B)(g_1(B) - g_2(B)) = \\ \sum_{A \in 2^X} m(B)(g_2(B) - g_1(B)) = \\ \left(1 - \sum\nolimits_{B \in 2^X} m(B) g_1(B)\right) - \left(1 - \sum\nolimits_{B \in 2^X} m(B) g_2(B)\right). \end{gathered}$$

We get that the expressions $\left(1 - \sum\nolimits_{B \in 2^X} m(B) g_i(B)\right)$, $i = 1,2$, can be considered as values of the linear imprecision index f if m obeys the condition 2) of Theorem 2. In this case we choose g_1 if $f(g_1) < f(g_2)$, i.e. the first source gives us more exact information than the second.

4 The algebraic structure of the set of linear imprecision indices

Let f_1, f_2 be linear functionals on M then their linear combination $f = af_1 + bf_2$, $a,b \in \mathbb{R}$ is also a linear functional. If we take into consideration set functions $\mu_{f_1}, \mu_{f_2}, \mu_f$, we see that $\mu_f = a\mu_{f_1} + b\mu_{f_2}$, i.e. the set of all linear functionals on M is a linear space and this space is isomorphic to the linear space M of all set functions on 2^X. It is easy to show that if f_1, f_2 are linear imprecision indices then their convex sum $f = af_1 + bf_2$, where $a,b \geq 0$, $a + b = 1$, is also linear imprecision index, i.e. the set of all linear imprecision indices is a convex set. We denote by M_I the set of all set functions μ_f, which correspond to linear imprecision indices on M_{low}. One can say that we understand the algebraic structure of a convex set if we find its extreme points. The following theorem gives the necessary and sufficient condition on an arbitrary $\mu \in M_I$ to be an extreme point.

Theorem 6. *Let* $\mu \in M_I$, $\mu = \sum_{A\in\mathfrak{B}} m(A)\overline{\eta}_{\langle A\rangle} - b\overline{\eta}_{\langle X\rangle}$, *where* $\mathfrak{B} \subseteq 2^X \setminus \{\emptyset, X\}$, $m(A) > 0$ *for all* $A \in \mathfrak{B}$, $b > 0$, *then* μ *is an extreme point of* M_I *iff functions* $\{1_A\}_{A\in\mathfrak{B}}$ *are linearly independent.*

Proof. Notice first that any $\mu \in M_I$ has the representation $\mu = \sum_{A\in\mathfrak{B}} m(A)\overline{\eta}_{\langle A\rangle} - b\overline{\eta}_{\langle X\rangle}$ by Corollary 1, $b > 0$, and $\mathfrak{B}$ is not empty. Secondly, $\mu(\{x\}) = 0$ for all $x \in X$, i.e.

$$\sum_{A\in\mathfrak{B}} m(A)1_A = b1_X .$$

We will show that μ is not an extreme point of M_I iff functions $\{1_A\}_{A\in\mathfrak{B}}$ are linearly dependent. This implies evidently the theorem statement. Assume that functions $\{1_A\}_{A\in\mathfrak{B}}$ are linearly dependent. Then there exist two different solutions of $\sum_{A\in\mathfrak{B}} \alpha_A 1_A = 1_X$ w.r.t. α_A, $A \in \mathfrak{B}$. We choose one of them as $\alpha_A^{(1)} = m(A)/b$, $A \in \mathfrak{B}$. Since $\alpha_A^{(1)} > 0$ for all $A \in \mathfrak{B}$, we can choose another solution $\alpha_A^{(2)}$ with $\alpha_A^{(2)} \geq 0$, $A \in \mathfrak{B}$. Let $b_2 = 1\Big/\left(\left(\sum_{A\in\mathfrak{B}} \alpha_A^{(2)}\right) - 1\right)$, then it is easy to see that $b_2 > 0$ and the set function μ_2, defined by

$$\mu_2 = \sum_{A\in\mathfrak{B}} b_2\alpha_A^{(2)}\overline{\eta}_{\langle A\rangle} - b_2\overline{\eta}_{\langle X\rangle},$$

is in M_I. Defining

$$c = \sup\left\{r \in \mathbb{R} \mid rb_2\alpha_A^{(2)} \leq m(A), A \in \mathfrak{B}, rb_2 \leq b\right\},$$

we confirm that $c \in (0,1)$, $\mu \geq c\mu_2$. Then

$$\mu_1 = \frac{1}{1-c}(\mu - c\mu_2) = \sum_{A\in\mathfrak{B}} m_1(A)\overline{\eta}_{\langle A\rangle} - b_1\overline{\eta}_{\langle X\rangle}.$$

where $m_1(A) = \frac{1}{1-c}\left(m(A) - cb_2\alpha_A^{(2)}\right)$, $b_1 = \frac{1}{1-c}(b - cb_2)$, is in M_I. We see that $\mu = (1-c)\mu_1 + c\mu_2$, i.e. we have proved that μ is not an extreme point of M_I.

Vice versa; assume that μ is not an extreme point of M_I. Then there exist set functions $\mu_1, \mu_2 \in M_I$ such that $\mu = a\mu_1 + b\mu_2$, where $a, b > 0$ and $a + b = 1$. Since $\mu_1, \mu_2 \in M_I(X)$ we have $\sum_{A\in\mathfrak{B}} m_i(A)1_A = b_i 1_X$, where $b_i > 0$, $i = 1,2$. Therefore, the equation $\sum_{A\in\mathfrak{B}} \alpha_A 1_A = 1_X$ has more than one solution w.r.t. $\alpha_A \in \mathbb{R}$, $A \in \mathfrak{B}$, hence, functions $\{1_A\}_{A\in\mathfrak{B}}$ are linearly dependent if μ is not an extreme point of M_I.■

Theorem 6 implies that the set M_I has the finite number of extreme points. According to the Theorem by Krein-Milman [9], any $\mu \in M_I$ can be represented as a convex sum of extreme points. However, it is a very hard problem to describe such extreme points explicitly. Further we consider one convex subset of M_I, for which this problem can be solved.

Definition 3. Let f be a linear imprecision index on M_{low}, then we call it *complementarily symmetrical* if $m^{\mu_f}(A) = m^{\mu_f}(\overline{A})$ for all $A \in 2^X \setminus \{\emptyset, X\}$.

Important examples of complementarily symmetrical linear imprecision indices are primitive imprecision indices. We see that

$$\nu_B(g) = g(X) - g(B) - g(\overline{B}) + g(\emptyset),$$

$$\begin{aligned}\mu_{\nu_B}(A) &= \eta_{\langle A\rangle}(X) - \eta_{\langle A\rangle}(B) - \eta_{\langle A\rangle}(\overline{B}) + \eta_{\langle A\rangle}(\emptyset)\\ &= \eta^{\langle\emptyset\rangle}(A) - \eta^{\langle\overline{B}\rangle}(A) - \eta^{\langle B\rangle}(A) + \eta^{\langle X\rangle}(A).\end{aligned}$$

Therefore, $m^{\mu_{\nu_B}}(A) = 1$ if $A \in \{\emptyset, X\}$, $m^{\mu_{\nu_B}}(A) = -1$ if $A \in \{B, \overline{B}\}$, and $m^{\mu_{\nu_B}}(A) = 0$ otherwise. We can also express μ_{ν_B} through plausibility functions. In this case

$$\begin{aligned}\mu_{\nu_B}(A) &= 1 - \eta_{\langle\overline{B}\rangle}(\overline{A}) - \eta_{\langle B\rangle}(\overline{A}) + \eta_{\langle X\rangle}(\overline{A})\\ &= \overline{\eta}_{\langle B\rangle}(A) + \overline{\eta}_{\langle\overline{B}\rangle}(A) - \overline{\eta}_{\langle X\rangle}(A).\end{aligned}$$

By Theorem 6 it is easy to show that primitive indices ν_B, $B \in 2^X \setminus \{\emptyset, X\}$, are extreme points of M_I. Actually, it follows from the fact that functions $\{1_B, 1_{\overline{B}}\}$ are linearly independent.

The role of primitive indices for describing the set of all complementarily symmetrical linear indices shows the following theorem.

Theorem 7. *The set of all complementarily symmetrical linear indices is convex. Any complementarily symmetri-*

cal linear index can be uniquely represented by a convex sum of primitive indices.

Proof. The convexity of all complementarily symmetrical linear indices it is obvious. Now we will prove that any complementarily symmetrical linear index can be represented by a convex sum of primitive indices. Let f be a complementarily symmetrical linear index and $g \in M_{low}$ then

$$f(g) = \sum_{B \in 2^X} m^{\mu_f}(B) g(B),$$

where $m^{\mu_f}(B) = m^{\mu_f}(\overline{B})$ for all $B \in 2^X \setminus \{\varnothing, X\}$. Let $\mathfrak{D} = \{B \in 2^X \setminus \{X\} \mid x \in B\}$, $\overline{\mathfrak{D}} = \{B \in 2^X \mid \overline{B} \in \mathfrak{D}\}$ for some $x \in X$ then $\mathfrak{D} \cup \overline{\mathfrak{D}} = 2^X \setminus \{\varnothing, X\}$, $\mathfrak{D} \cap \overline{\mathfrak{D}} = \varnothing$.

$$\begin{aligned} f(g) &= m^{\mu_f}(X) g(X) + m^{\mu_f}(\varnothing) g(\varnothing) \\ &\quad + \sum_{B \in \mathfrak{D}} m^{\mu_f}(B) \left(g(B) + g(\overline{B}) \right) \\ &= -\sum_{B \in \mathfrak{D}} m^{\mu_f}(B) \left(g(X) - g(B) - g(\overline{B}) + g(\varnothing) \right) \\ &\quad + \sum_{B \in \mathfrak{D}} m^{\mu_f}(B) \left(g(X) + g(\varnothing) \right) \\ &\quad + m^{\mu_f}(X) g(X) + m^{\mu_f}(\varnothing) g(\varnothing). \end{aligned}$$

We see that $\sum_{B \in \mathfrak{D}} m^{\mu_f}(B) = \sum_{B: x \in B} m^{\mu_f}(B) - m^{\mu_f}(X) = -m^{\mu_f}(X) = -1$. The equality $\sum_{B \in 2^X} m^{\mu_f}(B) = 0$ implies that $m^{\mu_f}(\varnothing) = -\sum_{B \in \mathfrak{D}} \left(m^{\mu_f}(B) + m^{\mu_f}(\overline{B}) \right) - m^{\mu_f}(X) = 1$. Hence,

$$f(g) = \sum_{B \in \mathfrak{D}} (-1) m^{\mu_f}(B) \nu_B,$$

where $(-1) m^{\mu_f}(B) \geq 0$ for all $B \in \mathfrak{D}$, and $\sum_{B \in \mathfrak{D}} (-1) m^{\mu_f}(B) = 1$, i.e. f can be represented by a convex sum of primitive indices.

We prove that the found representation is unique if we show that system $\{\nu_B\}_{B \in \mathfrak{D}}$ of all primitive indices is linearly independent in the linear space of all linear functionals on M, or we show the same property for set functions $\{\mu_{\nu_B}\}_{B \in \mathfrak{D}}$. It is easy to see that set functions $\mu_{\nu_B} = \overline{\eta}_{\langle B \rangle} + \overline{\eta}_{\langle \overline{B} \rangle} - \overline{\eta}_{\langle X \rangle}$, $B \in \mathfrak{D}$, are linearly independent, this follows immediately from the fact that set functions $\{\overline{\eta}_{\langle B \rangle}\}_{B \in 2^X \setminus \{\varnothing\}}$ are also linearly independent in M.■

Example 3. Let $\xi : X \to \mathbb{R}$, $\max_{x \in X} \xi(x) - \min_{x \in X} \xi(x) = 1$. Then we can define the linear imprecision index by Choquet integral [4] $f(g) = \int_X \xi d\overline{g} - \int_X \xi dg$, where $g \in M_{low}$. Then $\mu_f(B) = \max_{x \in B} \xi(x) - \min_{x \in B} \xi(x)$ for $B \neq \varnothing$. It is easy to show that such defined an index f is complementarily symmetrical. It is worth to mention that in the theory of imprecise probabilities $\int_X \xi d\overline{g}$ can be viewed as an upper estimate of the expectation $E[\xi]$, and $\int_X \xi dg$ as a lower estimate of the expectation $E[\xi]$.

Example 4. Let g be a coherent lower probability, and $g(A) = \min\{P_1(A), P_2(A)\}$, where $P_1, P_2 \in M_{\mathrm{Pr}}$, $A \in 2^X$. Let f be a complementarily symmetrical imprecision index. Then it is easy to show that

$$f(g) = \sum_{A \in 2^X} m(A) \left| P_1(A) - P_2(A) \right|,$$

where m is a non-negative set function on 2^X with $m(\varnothing) = 0$, $m(X) = 0$ and $\sum_{A \in 2^X} m(A) = 1$. Therefore, in this case we express the value of the imprecision index through the metric

$$d(P_1, P_2) = \sum_{A \in 2^X} m(A) \left| P_1(A) - P_2(A) \right|, \; P_1, P_2 \in M_{\mathrm{Pr}},$$

on M_{Pr} if m has the property $m(A) + m(\overline{A}) > 0$ for all $A \in 2^X \setminus \{\varnothing, X\}$.

5 The extension of imprecision indices to the set of all non-additive measures

In this section we will try to extend the notion of imprecision index. We consider first one simple generalization of imprecision indices onto the set M_{up}.

Definition 4. A functional $f : M_{up} \to [0,1]$ is called *imprecision index* if the following conditions are fulfilled: 1) $g \in M_{\mathrm{Pr}}$ implies $f(g) = 0$; 2) $f(g_1) \leq f(g_2)$ for all $g_1, g_2 \in M_{up}$ such that $g_1 \leq g_2$; 3) $f\left(\overline{\eta}_{\langle X \rangle}\right) = 1$.

We call an imprecision index f on M_{up} linear if it has linear properties on M_{up}. We can define this linear functional on the set of all set functions, and we take by definition that $f\left(\eta_{\langle \varnothing \rangle}\right) = 0$.

The following proposition shows the connection between imprecision indices on M_{low} and M_{up}.

Proposition 3. *Let* $f_1 : M_{low} \to [0,1]$ *then* f_1 *is an imprecision index on* M_{low} *iff the functional* $f_2 : M_{up} \to [0,1]$ *defined by* $f_2(g) = f_1(\overline{g})$, $g \in M_{up}$, *is an imprecision* *index on* M_{up}. *In addition,* f_1 *is a linear index on* M_{low} *iff* f_2 *is a linear imprecision index on* M_{up}.

Corollary 2. *Let* f *be a linear functional on* M *then* f *is an imprecision index on* M_{up} *iff the set function* μ^f, *defined by* $\mu^f(B) = f\left(\overline{\eta}_{\langle B\rangle}\right)$, *is in* M_I.

We see that using Proposition 3 and Corollary 2, we can formulate all results, proved for imprecision indices on M_{low}, through imprecision indices, defined on M_{up}. For example, Theorem 2 can be reformulated as follows.

Theorem 2*. *Any linear imprecision index* f *on* M_{up} *can be uniquely represented by*

$$f(g) = \sum_{B\in 2^X} m(B)g(B) - a,$$

where the set function m *obeys the following conditions:*
1) $m(\varnothing) = 0$, $m(X) = 0$, $m(B) \geq 0$ *for all* $B \in 2^X$;
2) $\sum_{B\in 2^X} m(B)1_B = 1_X$, $a = \sum_{B\in 2^X} m(B) - 1$.

Comparing Theorems 2 and 2*, we see that conditions 1), 2) are very close. If $a = 1$ then we can define an imprecision index on M_{low} and M_{up} by one linear functional. Namely, if the linear functional f defines the imprecision on M_{low}, then $-f$ defines the imprecision index on M_{up}, or $|f|$ defines an imprecision index on M_{low} and M_{up} simultaneously. In some cases, the sign of f may be useful, since it enables to check what the argument of f is: it is a lower or upper probability. If the argument g is not in $M_{low} \cup M_{up}$ we can say that g gives us rather lower estimations of probabilities than upper probabilities if $f(g) > 0$, and vice versa. In some cases, we should guarantee that $|f(g)| = |f(\overline{g})|$, in other words, the amount of imprecision is the same, if we describe uncertainty by lower or by upper probabilities. This situation is analyzed in the following proposition.

Proposition 4. *Let* f *be a linear functional on* M *and we use notations from Theorems 2, 2*. Then* $|f|$ *defines a linear imprecision index on* M_{low} *and* M_{up} *with* $|f(g)| = |f(\overline{g})|$ *for all* $g \in M_{low}$ *iff* f *is a complementarily symmetrical index on* M_{low}.

Proof. We see that $a = 1$ is the necessary condition, and this condition is fulfilled for complementarily symmetrical indices. Consider the sum

$$f(g) + f(\overline{g}) = 2 - \sum_{B\in 2^X} m(B)\overline{g}(B) - \sum_{B\in 2^X} m(B)g(B).$$

which has to be equal to zero for every $g \in M_{low}$.

$$f(g) + f(\overline{g}) = \sum_{B\in 2^X\setminus\{\varnothing,X\}} m(B)g(\overline{B}) - \sum_{B\in 2^X\setminus\{\varnothing,X\}} m(B)g(B) = \sum_{B\in 2^X\setminus\{\varnothing,X\}} \left(m(\overline{B}) - m(B)\right)g(B).$$

Since $m(\overline{B}) - m(B) = m^{\mu_f}(B) - m^{\mu_f}(\overline{B})$, for any complementarily symmetrical index $f(g) + f(\overline{g}) = 0$. We prove the proposition if we show that the condition $m(B) - m(\overline{B}) = 0$ is also necessary one. Let $g = \eta_{\langle D\rangle}$, $|D| = |X| - 1$ then $f(g) + f(\overline{g}) = m(\overline{D}) - m(D)$, i.e. $m(\overline{D}) - m(D) = 0$ for any $D \in 2^X$ with $|D| = |X| - 1$. Assume by induction the statement $m(\overline{D}) - m(D) = 0$ is true for any $D \in 2^X$ with $|D| = |X| - i$, $i = 1,...,k-1$, $k < |X| - 1$. We show that $m(\overline{D}) - m(D) = 0$ for any $D \in 2^X$, where $|D| = |X| - k$. Actually, choosing $g = \eta_{\langle D\rangle}$ with $|D| = |X| - k$, we get $f(g) + f(\overline{g}) = \sum_{D\subseteq B\subset X} \left(m(\overline{B}) - m(B)\right)g(B) = m(\overline{D}) - m(D)$, i.e. $m(\overline{D}) - m(D) = 0$ for any $D \in 2^X$ with $|D| = |X| - k$. ■

If we are going to generalize measuring imprecision for general case, i.e. imprecision indices are defined on M_{mon}, we should consider two types of uncertainty, caused by imprecision and inconsistency, and propose their interpretation. One possible interpretation consists in the following. Suppose that the set function $g \in M_{mon}$ should give us low estimates of probabilities, however, $g \notin M_{low}$. Then some of its values are greater than it is possible, and this implies that information contains some amount of inconsistency. Suppose that for measuring imprecision we use an index f on M_{low}. It seems to be logical to evaluate the amount of imprecision in g by the value

$$f_{\mathrm{Imp}}(g) = \inf_{q\in M_{low}|q\leq g} f(q).$$

We see that the functional f_{Imp} can be considered as an extension of f onto M_{mon}. Let $g \in M_{up}$ then $f_{\mathrm{Imp}}(g) = 0$, and we conclude that the amount of imprecision is equal to zero, i.e. we have exact information in our disposal, however, there is uncertainty caused by inconsistency. The amount of this uncertainty can be also evaluated. In this case we choose the same axiomatic for inconsistency index as for imprecision index for upper probabilities. Then we can measure inconsistency by

$f(\overline{g})$. If $g \in M_{mon}$ and $g \notin M_{up}$, we can introduce an inconsistency index by

$$f_{\text{Inc}}(g) = \inf_{q \in M_{up} | q \geq g} f(\overline{q}).$$

We see that $f_{\text{Inc}}(g) = 0$ if $g \in M_{low}$. It is clear that f_{Imp} is antimonotone on M_{mon}, i.e. $g_1 \leq g_2$ implies $f_{\text{Imp}}(g_1) \geq f_{\text{Imp}}(g_2)$ for $g_1, g_2 \in M_{mon}$. f_{Inc} is monotone on M_{mon}, i.e. $g_1 \geq g_2$ implies $f_{\text{Imp}}(g_1) \geq f_{\text{Imp}}(g_2)$ for $g_1, g_2 \in M_{mon}$. Further we will use the following notations: $g = \min\{g_1, g_2\}$ if $g(A) = \min\{g_1(A), g_2(A)\}$, for all $A \in 2^X$, $g, g_1, g_2 \in M_{mon}$. Next lemmas shows, how the problem of calculating f_{Imp} f_{Inc} can be simplified.

Lemma 1. $f_{\text{Imp}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\})$.

Proof. Since $\min\{\alpha, g\} \in M_{low}$ for any $\alpha \in M_{\text{Pr}}$, we conclude that $f_{\text{Imp}}(g) \leq \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\})$. Let $q \in M_{low}$, $q \leq g$ then there is an $\alpha \in M_{\text{Pr}}$ with $q \leq \alpha$. We see $q \leq \min\{\alpha, g\}$, i.e. $f_{\text{Imp}}(g) \geq \inf_{\alpha \in M_{\text{Pr}} | \alpha \leq g} f(\min\{\alpha, g\})$. So, there is one possibility $f_{\text{Imp}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\})$.■

The next result is proved analogously as Lemma 1.

Lemma 2. $f_{\text{Inc}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, \overline{g}\})$.

Lemma 3. Let $g = 0.5q + 0.5\overline{q}$, $q \in M_{mon}$, then $f_{\text{Imp}}(g) = f_{\text{Imp}}(\overline{g})$.

Proof. It is true because $g = \overline{g}$ in this case. ■

If we take another interpretation that $g \in M_{mon}$ gives us upper estimations of probabilities then we can follow the proposed scheme for defining imprecision and inconsistency indices, assuming that $\overline{g}$ gives us lower estimates of probabilities, i.e. if f is an imprecision index on M_{low}, then in this case $f_{\text{Imp}}(\overline{g})$ gives us the amount of imprecision, and $f_{\text{Inc}}(\overline{g})$ gives us the amount of inconsistency. In some situations we do not know what information we have in our disposal, we know only that g gives us estimates of probabilities, and we have to decide – it is lower estimates of probabilities or upper estimates of probabilities. One way, based on an imprecision index f, defined on M_{low}, consists in the following. We can assume that in the analyzed information the amount of imprecision should be greater or equal than the amount of inconsistency. Then, calculating the value

$$f_{\text{S}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\}) - \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, \overline{g}\}),$$

we suppose that g is rather lower probability than upper probability if $f_{\text{S}}(g) \geq 0$, and rather upper probability then lower probability if $f_{\text{S}}(g) < 0$.

Lemma 4. *Let f be a complementarily symmetrical linear imprecision index on M_{low} then $f_S(g) = f(g)$.*

Proof. Let all conditions of the lemma hold and $\mathfrak{D} = \left\{B \in 2^X \setminus \{X\} \mid x \in B\right\}$ for some $x \in X$ then by Theorem 7 $f(g) = \sum_{A \in \mathfrak{D}} m(A)\left(\overline{g}(A) - g(A)\right)$, where $g \in M_{mon}$, $m(A) \geq 0$ for all $A \in \mathfrak{D}$, and $\sum_{A \in \mathfrak{D}} m(A) = 1$. Let $\alpha \in M_{\text{Pr}}$, $g \in M_{mon}$ then

$$f(\min\{\alpha, g\}) - f(\min\{\alpha, \overline{g}\}) = \sum_{A \in \mathfrak{D}} m(A) q(A),$$

where $q(A) = \max\{\alpha(A), \overline{g}(A)\} - \min\{\alpha(A), g(A)\} - \max\{\alpha(A), g(A)\} + \min\{\alpha(A), \overline{g}(A)\} = \overline{g}(A) - g(A)$, i.e. $f(\min\{\alpha, g\}) = f(g) + f(\min\{\alpha, \overline{g}\})$, or

$$\inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\}) = f(g) + \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, \overline{g}\}),$$

and we get the result required. ■

Example 5. Let we have two source of information about the object of our interest in the form of possibility measures defined on the power set of the finite set X. These possibility measures are given by possibility distribution functions $\pi_i : X \to [0,1]$, $i = 1,2$, and values of the corresponding possibility and necessity measures Π_i, N_i, $i = 1,2$, are computed by formulas: $\Pi_i(A) = \max_{x \in X} \pi_i(x)$, $A \in 2^X \setminus \{\varnothing\}$, and $\Pi_i(\varnothing) = 0$; $N_i(A) = 1 - \Pi_i\left(\overline{A}\right)$, $A \in 2^X$. By our assumption, the values of N_i give us lower estimates of probabilities, the values of Π_i give us lower estimates of probabilities. For our example we assume that $X = \{x_1, x_2, x_3\}$, and functions $\pi_i : X \to [0,1]$, $i = 1,2$, are given by Table 1. Combining information of these two sources, we get the measure $g = \max\{N_1, N_2\}$, which should be a lower probability by our assumption, but it is not really in M_{low} because $g(A) > g(\overline{A})$ for $A = \{x_1\}$ and $A = \{x_2, x_3\}$ (see Table 2, where values of Π_i, $i = 1,2$, g, and corresponding dual measures are shown).

	x_1	x_2	x_3
π_1	1	0.5	0.5
π_2	0.4	1	0.6

Table 1: Values of possibility distribution functions.

x_1	x_2	x_3	Π_1	Π_2	N_1	N_2	g	$\overline{g}$
0	0	0	0	0	0	0	0	0
1	0	0	1	0.4	0.5	0	0.5	0.4
0	1	0	0.5	1	0	0.4	0.4	0.5
1	1	0	1	1	0.5	0.4	0.5	1
0	0	1	0.5	0.6	0	0	0	0.5
1	0	1	1	0.6	0.5	0	0.5	0.6
0	1	1	0.5	1	0	0.6	0.6	0.5
1	1	1	1	1	1	1	1	1

Table 2: Values of monotone measures.

Now for measuring imprecision and inconsistency we will use the following imprecision indices on $M_{low}(X)$:

$$v_1(g)=\left(2^{|X|}-2\right)^{-1}\sum\nolimits_{B\in 2^X}\left|\overline{g}(B)-g(B)\right|,$$

$$v_\infty(g)=\max\left\{\left|\overline{g}(B)-g(B)\right| \mid B\in 2^X\right\},$$

$$GH(g)=\frac{1}{\ln(X)}\sum_{B\in 2^X\backslash\{\varnothing\}} m_g(B)\ln|B|.$$

Notice that v_1, GH are linear imprecision indices, and v_∞ is non-linear one. The results of measuring uncertainty by these indices are shown in Table 3.

	Imprecision			Inconsistency		
	v_1	v_∞	GH	v_1	v_∞	GH
N_1	0.5	0.5	0.5	0	0	0
N_2	0.5(3)	0.6	0.526	0	0	0
g	0.2	0.5	0.2	0.03(3)	0.1	0.0288

Table 3: Evaluation of uncertainty by imprecision indices.

6 Summary and Conclusions

Although, measuring uncertainty plays a central role in various uncertainty theories, there is no possibility to find one true uncertainty measure. This can be explained by the fact that there are many various types of uncertainty, they have different interpretations; it is very difficult to understand their mutual interaction. One way for overcoming this problem is to find families of suitable uncertainty measures, satisfying some justified properties. The choice of the best uncertainty measure considerably depends on the problem solved. In this paper we have proposed how imprecision can be measured if uncertain information is described by monotone measures, in particular lower or upper probabilities. We have treated the case, where uncertainty consists of some randomness, imprecision, and inconsistency. The introduced axiomatics enables us to give detailed description of linear imprecision indices, and investigate some of them with symmetrical properties.

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