Imprecise probability methods for sensitivity analysis in engineering

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Abstract

This article addresses questions of sensitivity of output values in engineering models with respect to variations in the input parameters. Such an analysis is an important ingredient in the assessment of the safety and reliability of structures. A major challenge in engineering applications lies in the fact that high computational costs have to be faced. Methods have to be developed that admit assertions about the sensitivity of the output with as few computations as possible. This article serves to explore various techniques from imprecise probability that may contribute to achieving this goal.

Keywords. Reliability of structures, sensitivity analysis, random sets, fuzzy sets, simulation methods, aerospace engineering.

1 Introduction

The goal of this article is to demonstrate how various methods from imprecise probability theory can be employed in sensitivity analysis of engineering structures. We are motivated by a research project in aerospace engineering[1] which involves the determination of the buckling load of the frontskirt of the ARIANE 5 launcher under various loading and flight scenarios. The frontskirt is a reinforced light weight shell structure. The computation of the decisive parameter indicating failure, the load proportionality factor (LPF), is based on a finite element model[2]. Part of the project is to determine the most influential input parameters (loads, material constants, geometry) on the load proportionality factor in a sensitivity analysis. The goal is to evaluate the design and to assess the safety of the structure. The calculation of the output variable LPF – under a given single set of input parameters – takes about 32 hours on a high performance computer. In addition to the extremely high computational cost, the LPF may depend in a non-differentiable manner on some of the input parameters, especially variations in the geometry. A classical sensitivity analysis of the complete structure is currently out of reach.

Engineering information on the variability of the input parameters usually consists of a central value and a coefficient or range of variation. The basic strategy for arriving at a sensitivity assessment will be to successively freeze the input parameters and study the effect on the variability of the output. We wish to do this without artificial parametric assumptions and with as few calls of the finite element program as possible. We will explore the usability of methods from imprecise probability theory for this purpose. In particular, we shall model the input variability by means of

– random sets and Tchebycheff's inequality;

– fuzzy sets and Hartley-like measures;

– intervals and sampling from a Cauchy distribution;

– standard Monte-Carlo simulation and resampling.

A detailed description of the respective methods will follow in four sections, with a final section devoted to a comparison of the methods. The question of modelling correlations between the input variables will be addressed in the appropriate sections. We shall exemplify the results with the aid of a simplified finite element model simulating part of a space craft launcher (Figure 1). The computational cost for the simplified model is one hour per call of the program.

In the sensitivity analysis, up to 17 input parameters were taken into account. A tentative description of the meaning of the parameters as well as their nominal values can be read off from Table 1.

[1] ICONA-project, Intales GmbH Engineering Solutions and University of Innsbruck, supported by TransIT Innsbruck and by EADS Astrium ST.

[2] The load proportionality factor is defined as the limiting value in an incremental procedure, in which the dynamic loads during a flight scenario are increased stepwise until breakdown of the structure is reached.

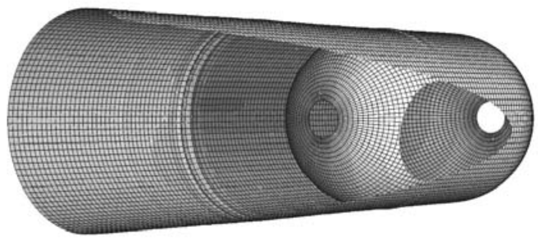

Figure 1: Simplified finite element model.

For background material on sensitivity analysis we refer to the Special Issue [9], in particular the survey article [10] and to [8], for random sets, to [18, 19], for random and fuzzy sets, to [5, 14], for probability boxes, to [3], for a review on probabilistic treatment of uncertainty in structural engineering as well as information on variability of typical input parameters, to [24].

i	Parameter X_i	Mean μ_i
1	Initial temperature	293 K
2	Step1 thermal loading cylinder1	450 K
3	Step1 thermal loading cylinder2	350 K
4	Step1 thermal loading cylinder3	150 K
5	Step1 thermal loading sphere1	150 K
6	Step1 thermal loading sphere2	110 K
7	Step2 hydrostatic pressure cylinder3	0.4 MPa
8	Step2 hydrostatic pressure sphere1	0.4 MPa
9	Step2 hydrostatic pressure sphere2	0.4 MPa
10	Step3 aerodynamic pressure	−0.05 MPa
11	Step4 booster loads y-direction node1	40000 N
12	Step4 booster loads y-direction node2	20000 N
13	Step4 booster loads z-direction node1	3.e6 N
14	Step4 booster loads z-direction node2	1.e6 N
15	Step4 mechanical loads x-direction	100 N
16	Step4 mechanical loads y-direction	50 N
17	Step4 mechanical loads z-direction	300 N

Table 1: Description of input parameters no. 1 − 17.

2 Random set methods

It has been argued in [20, 21] that random intervals constructed by Tchebycheff's inequality can serve as a non-parametric model of the variability of a parameter, given its mean value and variance as sole information. We begin with the univariate case of a real-valued random variable X. Let $\mu = \mathrm{E}(X)$ be its expectation and $\sigma^2 = \mathrm{V}(X)$ be its variance. Tchebycheff's inequality asserts that

$$P\big(|X-\mu| > d_\alpha\big) \le \alpha, \qquad d_\alpha = \sigma/\sqrt{\alpha}. \qquad (1)$$

Equipping the unit interval $(0,1]$ with the uniform probability distribution, the non-parametric confidence intervals

$$I_\alpha = [\mu - d_\alpha, \mu + d_\alpha], \quad \alpha \in (0,1] \qquad (2)$$

define a random set. By construction, the following formulas for the belief in the set I_α and the plausibility of its complement I_α^{c} hold:

$$\begin{aligned} \underline{P}(I_\alpha) &= \textstyle\int_{\{\beta\in(0,1]:I_\beta\subset I_\alpha\}} \mathrm{d}\beta = 1-\alpha \le P(I_\alpha),\\ \overline{P}(I_\alpha^{\mathsf{c}}) &= \textstyle\int_{\{\beta\in(0,1]:I_\beta\cap I_\alpha^{\mathsf{c}}\neq\emptyset\}} \mathrm{d}\beta = \alpha \ge P(I_\alpha^{\mathsf{c}}). \end{aligned}$$

This shows that the random set description provides a conservative assessment of the variability X. In applications, the range of the parameter X may be confined to a compact interval $[x_{\min}, x_{\max}]$. In this case, the random set will be truncated to

$$I_\alpha = [(\mu - d_\alpha) \vee x_{\min}, (\mu + d_\alpha) \wedge x_{\max}].$$

In the multivariate case $X = (X_1, \dots, X_d)$ where each parameter X_i is modelled as a random set as in (2), we form the joint random set (assuming random set independence)

$$\alpha = (\alpha_1, \dots, \alpha_d) \to A_\alpha = I^1_{\alpha_1} \times \dots \times I^d_{\alpha_d}$$

again with the uniform distribution on the probability space $(0,1]^d$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. If the input variables $X = (X_1, \dots, X_d)$ are modelled as a random set A_α, $\alpha \in (0,1]^d$ (equipped with the uniform probability distribution), the output variable is given by the random set $g(A_\alpha)$, $\alpha \in (0,1]^d$. A visualization of the output can be obtained by means of the upper and lower distribution functions (or *probability box*, [3])

$$\begin{aligned} \overline{F}(x) &= P\big(\alpha : g(A_\alpha) \cap (-\infty, x] \neq \emptyset\big)\\ \underline{F}(x) &= P\big(\alpha : g(A_\alpha) \subset (-\infty, x]\big). \end{aligned} \qquad (3)$$

In the numerical evaluation, the joint random set is approximated by a finite random set with focal elements

$$I^1_{\alpha_1} \times \dots \times I^d_{\alpha_d}, \qquad \alpha_j \in \{\tfrac{1}{n}, \tfrac{2}{n}, \dots, 1\},$$

each with probability weight n^{-d}. The input-output function is evaluated as follows: First, an interval $Q \subset \mathbb{R}^d$ is determined that bounds the relevant range of the input variables X. Next, the values of the function g are computed at the m^d nodes of a uniform grid on Q. The output $g(Q)$ is approximated by a response surface $\hat{g}(Q)$ obtained by multilinear splines. More precisely, to compute the image of one of the sets A_α, $\hat{g}(Q)$ is evaluated at all grid points inside A_α and all points on its edges intersecting one of the grid lines. The interval $g(A_\alpha)$ is approximated by the minimum and maximum value thus obtained. Finally, the probability box (3) is calculated by adding the weights when appropriate. The essential computational effort thus amounts to m^d calls of the finite element program.

Figure 2 shows the result of the calculation of the load proportionality factor (LPF) where the three input parameters X_3, X_{13}, X_{14} (temperature cylinder 2, booster load node 1 in z-direction, booster load node 2 in z-direction) were kept variable. The variance σ for the Tchebycheff model was adjusted such that the base intervals $[x_{\min}, x_{\max}]$ for each of the parameters was symmetric around the corresponding mean μ with spread $\pm 0.15\mu$. In this case, $d = 3$ and we chose $m = 5$ so that 125 calls to the FE-program were required.

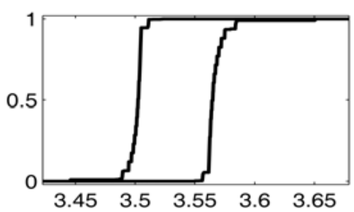

Figure 2: Probability box: LPF, 3 input variables.

Example 1 To assess the sensitivity of the load proportionality factor LPF with respect to the parameters X_3, X_{13}, X_{14} we again use the Tchebycheff model for each of the parameters with spread 0.15 times their mean values. Then we successively set one of the resulting $\sigma_3, \sigma_{13}, \sigma_{14}$ equal to zero (while keeping the others at their given value), go through the calculation indicated above and plot the resulting probability box (solid lines − the thin lines indicate the unperturbed result from Figure 2). This is displayed in Figure 3 and shows that setting $\sigma_{13} = 0$ produces the biggest reduction of the width of the probability box, while setting $\sigma_{14} = 0$ has little effect. We infer that the parameter X_{14} has the least influence on the variability of the response, while X_{13} exerts the biggest influence.

The pinching strategy in the case of probability boxes is further explicated in [4] and applied in [21]. Questions of dependence or interactivity of the input variables are left aside in this section. Dependence could be modelled by copulas on the underlying probability space $(0, 1]^d$ or by restrictions on the set of probability measures on $\mathbb{R}^d$ defined by the random set.

3 Fuzzy sets

In this section, one-dimensional input variables will be modelled as *normalized fuzzy numbers*, that is as fuzzy subsets B of the real line with upper semi-continuous

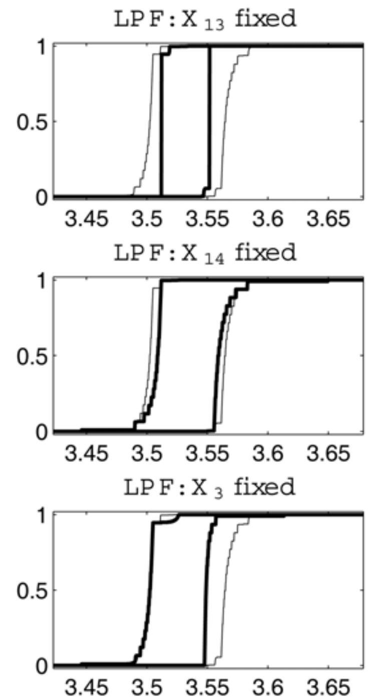

Figure 3: Probability box: LPF, frozen variables.

membership function $\pi_B(x)$ that attains the value 1. The α-level set of B is the set

$$B_\alpha = \{x \in \mathbb{R} : \pi_B(x) \geq \alpha\}, \quad \alpha \in (0, 1].$$

In the multivariate case, the non-interactive joint fuzzy set is defined as follows. Given d univariate fuzzy sets $B^1, \ldots, B^d$, the joint fuzzy set has the α-level sets

$$B_\alpha = B_\alpha^1 \times \ldots \times B_\alpha^d, \quad \alpha \in (0, 1].$$

Interactivity will be modelled by certain parametric restrictions on the α-level sets. To avoid combinatorial complications, we shall treat interactivity of at most two out of the d variables. Since an α-level set of the form $B_\alpha^i \times B_\alpha^j$ is a homothetic image of the unit square, it suffices to give the definitions for $B_\alpha^1 = B_\alpha^2 = [0, 1]$. Following [27], interactivity will be modelled by replacing the unit square by a diamond-shaped region, symmetric around one of the diagonals. Let $0 \leq \rho \leq 1$ and define the points $P_1, \ldots, P_4$ by

$$\begin{aligned} P_1 &= (\rho/2, \rho/2), & P_2 &= (1 - \rho/2, \rho/2), \\ P_3 &= (1 - \rho/2, 1 - \rho/2), & P_4 &= (\rho/2, 1 - \rho/2). \end{aligned}$$

Interactivity of *positive degree* ρ is modelled by taking the rhombus with corners $\{(0, 0), P_2, (1, 1), P_4\}$ as joint level set, while interactivity of *negative degree* $-\rho$ is modelled by the rhombus with corners $\{(0, 1), P_1, (1, 0), P_3\}$ as joint level set (Figure 4).

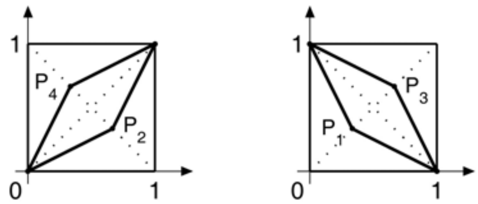

Figure 4: Positive/negative interactivity.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. If the input variables $X = (X_1, \ldots, X_d)$ are modelled as a non-interactive or interactive fuzzy set with α-level sets B_α as above, Zadeh's extension principle yields the output variable as the fuzzy number with level sets $g(B_\alpha)$, $\alpha \in (0, 1]$.

While a fuzzy set can be interpreted as a random set (cf. e. g. [5]) and the procedure appears similar to the one of Section 2, there is a fundamental difference in the multivariate case: in fuzzy set theory, only α-level sets of the same level are combined to produce the joint fuzzy set, while for random sets, the focal elements are obtained as products with respect to any combination and thus are indexed by the product space $(0, 1]^d$.

Example 2 In the assessment of the sensitivity of the load proportionality factor LPF with respect to the input parameters X_3, X_{13}, X_{14}, these parameters were modelled as symmetric triangular fuzzy numbers, with central values μ_i from Table 1 and spread $\pm 0.15\mu_i$ as before. The numerical calculation is based on the response surface method explained in Example 1. The images of the α-level sets are again computed by piecewise multilinear combination. To handle possible lack of monotonicity of the function g, we start with level $\alpha = 1$ and go the way down to $\alpha = 0$, insuring at each step that the approximations satisfy $g(A_\beta) \subset g(A_\alpha)$ for $\alpha < \beta$.

In the non-interactive case, the procedure for determining the sensitivity of the output with respect to the input variables is the same as in Example 1. The initial calculation is performed with proportional spreads $\pm 0.15\mu_i$. Then we successively replace one of the triangular fuzzy numbers by its crisp central value μ_i, and compute the output as a fuzzy number. The result gives a good visual representation of the change of variability. This can be quantified using e. g. the Hartley-like measure

$$\mathrm{HL}(B) = \int_0^1 \log\big(1 + \lambda(B_\alpha)\big)\, \mathrm{d}\alpha$$

of fuzzy sets B as proposed by [14] (see also [1] for further implementation of this idea in sensitivity analysis and [6] for interval-valued indices). The result is depicted in Figure 5, where the outer contour is the membership function of the fuzzy LPF with all input parameters fuzzy, while the shaded region is bounded by the membership function of the fuzzy LPF with successively frozen input parameters. It confirms the observations obtained by the random set method: X_{13} is the most influential parameter, followed by X_3 and then X_{14}. This can be explained by the model set-up: X_{13} refers to a large booster load on one side of the frontskirt, while X_{14} signifies a much smaller booster load on the opposite side. The Hartley-like measures displayed in Table 2, though, show that some, albeit small, influence of parameter X_{14} is detectable.

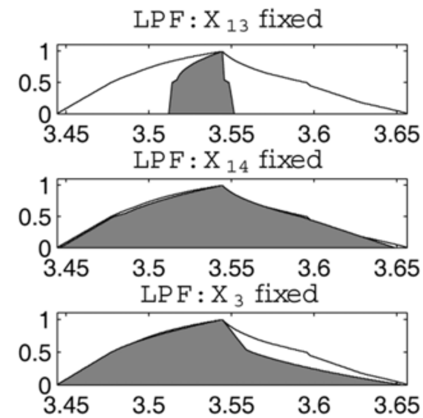

Figure 5: Fuzzy sets: LPF, frozen variables, noninteractive case.

Fuzzy set	HL-measure
no fixing	0.1481
X_{13} fixed	0.0398
X_{14} fixed	0.1430
X_3 fixed	0.1268

Table 2: Hartley-like measures of outputs, non-interactive input.

Example 3 This example serves to show how the effect of possible correlations between two of the input parameters on the sensitivity can be assessed. *Correlation* will be interpreted here as degree of interactivity as described above. In this example, we assume a degree of interactivity $\rho = 0.98$ between parameters X_{13} and X_{14}. The remaining parameters are treated as non-interactive. The α-level sets are of cylindrical shape with a rhombic base R_α, say. Their images are again computed by piecewise multilinear combination. Otherwise, the procedure of successively freezing variables is similar: For example, when X_{13} is frozen at its central value μ_{13}, the interactivity restricts X_{14} to vary along the intersection of R_α with the line through

μ_{13} parallel to the x_{14}-axis, while X_3 varies in its original α-level interval.

The result is shown in Figure 6; the meaning of the contour and the shaded region is the same as in Figure 5. The outcome confirms the prominence of pa-

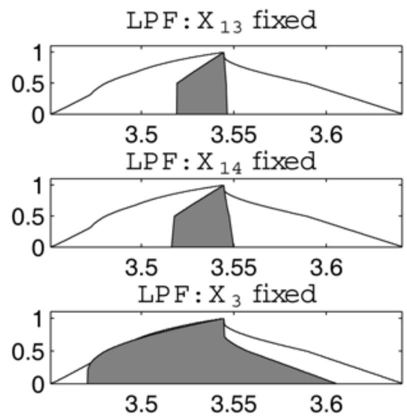

Figure 6: Fuzzy sets: LPF, frozen variables, interactive case.

rameter X_{13}; as a consequence of the correlation, parameter X_{14} is seen to exert a comparable influence. The result also demonstrates that the correlation changes the sensitivity of the output with respect to parameter X_3. Table 3 shows the Hartley-like measures of the fuzzy output under successive freezing of input variables. One may note that the study of the influence of correlations can be implemented in the fuzzy approach with ease.

Fuzzy set	HL-measure
no fixing	0.1357
X_{13} fixed	0.0287
X_{14} fixed	0.0329
X_3 fixed	0.1011

Table 3: Hartley-like measures of outputs, interactive input.

As in Example 1, the computational effort using the response surface consisted in 125 calls of the finite element program. The vertical jumps of the membership function in Figure 6 indicate that the output does not depend monotonically on the input variables. Closer inspection (done by producing an array of two-dimensional plots of the partial maps $X_i \to$ LPF) showed that this is indeed the case. Therefore, the accuracy of the method using just 125 grid values is in question. A number of additional explicit evaluations showed that the accuracy of the boundaries of the α-level sets for the LPF is in the range of ± 0.02 in absolute value.

4 Interval bounds

This section is devoted to interval estimates of input and output parameters. Suppose that the variability of each input parameter X_i is described by an interval $[\mu_i - \Delta_i, \mu_i + \Delta_i]$ of spread Δ_i around a central value μ_i. It has been argued in [16], that an estimate of the output interval can be obtained by Monte Carlo simulation using the Cauchy distribution.

The underlying theory from [16] is as follows. Suppose we wish to estimate the difference

$$\Delta y = g(x_1, \ldots, x_d) - g(\mu_1, \ldots, \mu_d)$$

where $|\Delta x_i| = |x_i - \mu_i| \leq \Delta_i$. Linearization around the mean value gives

$$|\Delta y| \leq \Delta = \sum_{i=1}^{d} |c_i| \Delta_i, \qquad c_i = \frac{\partial g}{\partial x_i}(\mu_1, \ldots, \mu_d).$$

If the X_i are independent random variables following a Cauchy distribution with scale parameter Δ_i, then $Y = c_1 X_1 + \ldots + c_d X_d$ obeys a Cauchy distribution with scale parameter Δ. This offers the possibility of computing the bound Δ on the output spread by Monte Carlo simulation.

The algorithm runs along the following lines. To produce a single realization, a d-dimensional sample $(z_1, \ldots, z_d)$ of Cauchy distributed variables with scale parameters 1 is taken. Setting $K = \max_{1 \leq i \leq d} |z_i|$, one has that $\delta_i = \Delta_i z_i / K$ has a Cauchy distribution with scale parameter Δ_i / K. Putting $x_i = \mu_i + \delta_i$ it follows that

$$Z = K\big(g(x_1, \ldots, x_d) - g(\mu_1, \ldots, \mu_d)\big)$$

is a realization of a Cauchy distributed variable with desired scale parameter Δ (this is true exactly when g is linear and otherwise approximately). An n-fold repetition yields the Monte Carlo sample of size n of the variable Z. Fitting a Cauchy distribution – e. g. by the maximum likelihood method – produces an estimate of the spread Δ of the output interval $[g(\mu_1, \ldots, \mu_d) - \Delta, g(\mu_1, \ldots, \mu_d) + \Delta]$. The computational effort for this estimate is n calls of the finite element program and thus independent of the dimension d. This offers the possibility to include a larger number of input variables in the analysis.

Example 4 In this calculation, 17 input parameters were included with nominal values displayed in Table 1. The spreads Δ_i were taken as 0.15-times the nominal values μ_i. We used a direct Monte Carlo method to produce a sample of size $n = 100$. The value of the load proportionality factor LPF was obtained as $\mu = g(\mu_1, \ldots, \mu_d) = 3.5443$. The simulation resulted in an estimate for its spread of $\hat{\Delta} = 0.2924$.

In the next step, the distribution of the resulting spread Δ was estimated by resampling. We employed 10000 random subsamples of size 100 (with repetition), following the suggestions in [23]. This resulted in a 95%-confidence interval for Δ of $CI_{0.95}(\hat{\Delta}) = [0.2281, 0.3685]$. The essential computational effort consisted in $n = 100$ calls of the finite element program.

Remark 5 A sensitivity analysis could be based on this method, again by freezing variables successively. It is possible to reduce computational cost by using the same Monte Carlo sample and approximating the frozen variables by a truncated Cauchy distribution. More precisely, instead of setting $\Delta_1 = 0$, say, we select the random numbers $(x_2, \ldots, x_d)$ computed above from the part of the population $(x_1, x_2, \ldots, x_d)$ which satisfies $|\delta_1| < \varepsilon$ for a suitably chosen small ε. This is justified, because the resulting truncated $(d-1)$-dimensional random variables converge in distribution to the ones with Δ_1 frozen at the value 0 as $\varepsilon \to 0$. However, successive simultaneous freezing of two or more variables requires repeated Monte Carlo simulation because the sample size would be too small for repeated truncation.

A more troublesome observation concerns the accuracy of the Cauchy method in our situation where the output function g is a nonlinear finite element computation resulting in the LPF. It turned out that the simulations of the auxiliary variable Z actually failed the KS-test for being Cauchy distributed. This means that our output function g is too far away from linearity and thus puts the accuracy of the Cauchy method into question in this context.

5 Monte Carlo simulation

To complete the analysis, we have a glimpse at direct Monte Carlo simulation in sensitivity analysis. Methods like scatterplots (input – output) and computing the weighted contribution of each input variable to the variance of the output are commonplace and will not be discussed here. These methods suffer the problem that hidden interactions may have a significant effect on the decomposition of the variance (see, however, [2]). We therefore turn to a method which intends to remove the influence of co-variates on the correlation between a given input variable X_i and the output variable Y. This method is based on the partial rank correlation coefficient (PRCC).

We recall that partial correlation between two random variables X_i and Y given a set of co-variates $X_{\smallsetminus i} = \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d\}$ is defined as the correlation between the two residuals $e_{X_i \cdot X_{\smallsetminus i}}$ and $e_{Y \cdot X_{\smallsetminus i}}$ obtained by regressing X_i on $X_{\smallsetminus i}$ and Y on $X_{\smallsetminus i}$, respectively. More precisely, one first constructs the two regression models

$$\widehat{X}_i = \alpha_0 + \sum_{j \neq i} \alpha_j X_j, \quad \widehat{Y} = \beta_0 + \sum_{j \neq i} \beta_j X_j,$$

obtaining the residuals

$$e_{X_i \cdot X_{\smallsetminus i}} = X_i - \widehat{X}_i, \quad e_{Y \cdot X_{\smallsetminus i}} = Y - \widehat{Y}.$$

Since $e_{X_i \cdot X_{\smallsetminus i}}$ and $e_{Y \cdot X_{\smallsetminus i}}$ are those parts of X_i and Y that remain after subtraction of the best linear estimates in terms of $X_{\smallsetminus i}$, the partial correlation coefficient

$$\rho_{X_i, Y \cdot X_{\smallsetminus i}} = \rho(e_{X_i \cdot X_{\smallsetminus i}}, e_{Y \cdot X_{\smallsetminus i}})$$

quantifies the linear relationship between X_i and Y after removal of any part of the variation due to the linear influence of $X_{\smallsetminus i}$. Applying a rank transformation to the variables X_i and Y leads to the partial rank correlation coefficient (PRCC). For further background on PCCs and PRCCs, see [7, 11, 22].

Example 6 To estimate the influence of each of the 17 input parameters from Table 1 on the output LPF, we performed a Monte Carlo simulation of size $n = 100$ with uniformly distributed input variables (on the intervals as in Example 4), using Latin hypercube sampling, an efficient stratified sampling strategy.

To obtain a sample of size n, the Latin hypercube sampling plan divides the range of each variable X_i into n disjoint subintervals of equal probability. First, n values of each variable X_i, $i = 1, \ldots, d$, belonging to the respective subintervals are randomly selected. Then the n values for X_1 are randomly paired without replacement with the n values for X_2. The resulting pairs are then randomly combined with the n values of X_3 and so on, until a set of n d-tuples is obtained. This set forms the Latin hypercube sample. The advantage of Latin hypercube sampling is that sampled points are evenly distributed through design space, thereby covering regions possibly important for the input-output map which might be missed by direct Monte Carlo simulation. It can be shown that the variance of an estimator based on Latin hypercube sampling is asymptotically smaller than the variance of the direct Monte Carlo estimator, and possibly markedly smaller when the input-output map is partially monotonic [8, 17, 26].

For additional accuracy in view of the rather small sample size we subjected the simulated variables to correlation control (see [12, 13]). This procedure consists in a rearrangement of the originally simulated values such that the resulting empirical rank correlation matrix is close to diagonal.

The resulting PRCCs can be seen in Figure 7. For further statistical confirmation, we performed a resampling procedure as in Example 4, producing bootstrap confidence intervals for the partial rank correlation coefficients as displayed in Figure 8. Accordingly, only the PRCCs of the parameters X_1, X_3, X_9, X_{13} and X_{14} test to be nonzero.

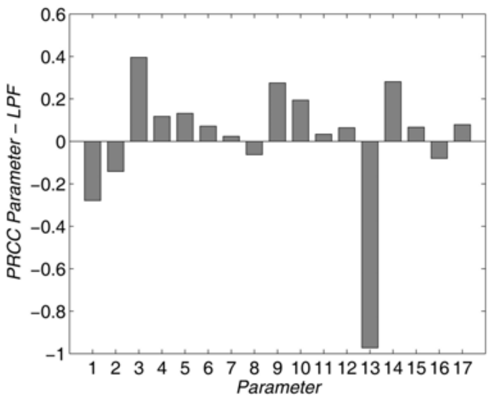

Figure 7: Partial rank correlation coefficients.

The outcome confirms the results of the sensitivity analysis in the previous sections: Among the parameters X_3, X_{13} and X_{14}, the one with the biggest influence is X_{13}, followed by X_3 and X_{14}.

We also ran various tests with correlated input as in Example 3 which confirmed the observed sensitivities. However, each test required a new Monte Carlo simulation with sample size $n = 100$. In addition, we computed Sobol indices [25] for groups of variables; this, however, again requires additional Monte Carlo simulations.

6 Summary and Conclusions

Starting from a research project in aerospace engineering one of whose goals was to determine the sensitivity of the buckling load of the frontskirt of the ARIANE 5 launcher with respect to certain input parameters, we explored various methods from probability and imprecise probability theory. In view of the excessive computational costs of a single run of the finite element program, the major challenge was to develop methods with as few calls of the program as possible. We used a simplified model of the launcher for the numerical tests of the methods.

The methods under scrutiny were random sets and Tchebycheff's inequality, fuzzy sets and Hartley-like measures, intervals and sampling from a Cauchy distribution, standard Monte-Carlo simulation and resampling. Criteria for the evaluation are

– computational effort

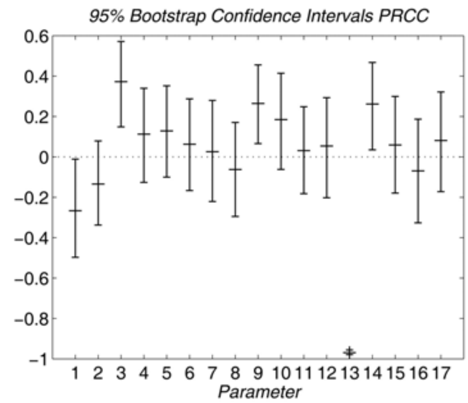

Figure 8: Partial rank correlation coefficients, confidence intervals.

– applicability to large scale problems

– accuracy

– avoidance of tacit assumptions

– reliability and clarity of interpretation

– possibility of analyzing correlated input.

Generally speaking, the Monte Carlo simulation methods are computationally least expensive. For our sensitivity study, a sample size of $n = 100$ appeared sufficient. In addition – as is well known – the sample size can be chosen independently of the number of input variables, so that we could include all 17 variables in our study. These methods are clearly applicable to large scale problems. Disadvantages are that parametric assumptions on the input variables have to be made and that freezing of variables requires repetition of the $n = 100$ simulations. Thus computing PRCCs plus resampling is possible irrespective of the problem scale, but variance decomposition by freezing variables is not. The same applies to analyzing sensitivity with respect to input correlations, which requires repetition of the simulation as well. The numerical accuracy of the Monte Carlo simulation is well known to be of order $1/\sqrt{n}$ times the standard deviation of the simulated variable. In view of the coefficients of variation which were in the range of 10% this appeared sufficient for the sensitivity study.

We emphasize that the results of a Monte Carlo simulation are amenable to resampling, which introduces little additional computational effort (no further evaluations of the costly input-output map are needed). In this way, bootstrap confidence intervals can be obtained that may serve as statistical estimates of the accuracy of the results. For example, we estimated the bias of each partial rank correlation coefficient, that is, the absolute value of the difference of the mean of the resampled data and the initial estimate. The estimated bias resulted to be less than 2% of the initial estimate. Further, the significance of the resulting ranking of the influence of the respective input parameters can be assessed by comparing the bootstrap confidence intervals.

The Cauchy method is a simulation method for estimating the spread of the output interval. The resulting estimate is non-parametric in as much as only the spreads of the input variables enter. As a subcase of Monte Carlo simulation, everything that has been said above applies here as well. A problematic point is that the method is derived under the assumption that the output function is approximately linear. In our case, the output function is substantially nonlinear. By means of repeated simulations we observed a quite substantial lack of accuracy of the estimate of the output spread in our case. Namely, direct Monte Carlo simulations of size $n = 100$ of the output variable LPF, with uniformly distributed input variables, produced an output range of $[3.45, 3.65]$. This indicates that the range was largely overestimated by the Cauchy method (see Example 4). This could possibly be overcome by the suggestion of [16] of repeated bisection of the input interval, though at an increase in computational cost.

Both in the fuzzy set and random set methods, the output α-level sets and focal sets, respectively, are computed by searching for the maximum and minimum of the corresponding output range. Sufficient accuracy can only be obtained by a larger number of calls of the output function, evaluated on a grid of input data. In addition, the grid size increases exponentially with the number of input variables. These methods appear feasible only in the case of medium size problems and a small number of input variables. Monotonicity or partial monotonicity of the output function increases accuracy and helps reducing the number of computations required.

Test runs with finer grids showed that the numerical error of the interpolation (i. e. replacing the true output function by a piecewise bilinear response surface) was less than 1%, thus definitely satisfactory. However, the optimization error introduced when calculating the boundaries of the output level sets turned out to be about ± 0.02 in absolute value, which is around 10 - 20% of the spread of the base level (see end of Section 3).

The numerical error in the boundaries of the output level sets appears less influential in the random set method. This is due to a certain averaging effect. Indeed, in the fuzzy model the computation of ℓ output level sets corresponds to ℓ input level sets, whereas in the random set model – at least when using random set independence – a combination of ℓ^d input focal sets enters (d the number of variables).

Both methods are essentially non-parametric. The random set model we used is generated by Tchebycheff's inequality and hence non-parametric by definition. In the fuzzy set model, we used triangular fuzzy numbers as input. These can be seen as a collection of intervals of linearly changing length. The α-level sets resulting from the computation determine the output range when the input varies over d-dimensional intervals of length proportional to $1 - \alpha$.

The fuzzy model in combination with the response surface technique has an additional advantage: it allows the a-posteriori introduction of interactivity between the input variables without the need for new calls of the output function. The effect of interactive input can simply be evaluated by interpolation in the response surface.

We finally comment on the practicality of upscaling to the full problem. This remains a major challenge. The computational structure of the given problem consists in a nonlinear, incremental procedure. The LPF is obtained as the ultimate load value beyond which the computed solution cannot be prolonged. This may be either due to a bifurcation point or to a breakdown of the structure. We currently pursue two strategies. One strategy is a perturbation method that replaces the full model by a quadratic approximation when a bifurcation point is reached. This is based on Koiter's asymptotic analysis of post-buckling of shells, see e. g. [15]. The sensitivity analysis would be done with the asymptotic model in place of the full model. The second strategy is to start the sensitivity analysis at a later stage of the iterative procedure. Both methods require to access the finite element code at a deeper level. A certain difficulty which we expect to encounter stems from the fact that the incremental procedure is path dependent. Thus varying the input parameters late in the process could be misleading, as initial variations might result in a quite different path to breakdown.

Acknowledgements

We are grateful to Herbert Haller for providing the finite element models and advice on engineering questions. Thanks are also due to Christof Neuhauser and Alexander Ostermann for discussions on numerical problems, to Hermann Starmans for the project management and to Robert Winkler for help with the finite element codes.

References

[1] D. A. Alvarez. Nonspecifity for infinite random sets. Preprint, Unit for Engineering Mathematics, University of Innsbruck, 2006.

[2] G. E. B. Archer, A Saltelli, I. M. Sobol. Sensitivity measures, ANOVA-like techniques and the use of the bootstrap. *Journal of Statistical Computation and Simulation*, 58:99–120, 1997.

[3] S. Ferson, V. Kreinovich, L. Ginzburg, D. S. Myers and K. Sentz. Constructing probability boxes and Dempster-Shafer structures. *SANDIA Report* SAND2002-4015, Sandia National Laboratories, Albuquerque, 2003.

[4] S. Ferson and W. T. Tucker. Sensitivity analysis using probability bounding. *Reliability Engineering & System Safety*, 91:1435-1442, 2006.

[5] I. R. Goodman, H. T. Nguyen. Fuzziness and randomness. In: C. Bertoluzza, M. Á. Gil, D. A. Ralescu, editors. *Statistical Modeling Analysis and Management of Fuzzy Data*. Physica-Verlag, Heidelberg, 2002.

[6] J. W. Hall. Uncertainty-based sensitivity indices for imprecise probability distributions. *Reliability Engineering & System Safety*, 91:1443-1451, 2006.

[7] J. C. Helton, F. J. Davis. Sampling-based methods for uncertainty and sensitivity analysis. *SANDIA Report* SAND99-2240, Sandia National Laboratories, Albuquerque, 2000.

[8] J. C. Helton, F. J. Davis. Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems. *SANDIA Report* SAND2001-0417, Sandia National Laboratories, Albuquerque, 2002.

[9] J. C. Helton, R. M. Cooke, M. D. McKay, A. Saltelli, editors. *The Fourth International Conference on Sensitivity Analysis of Model Output (SAMO 2004), Santa Fe, New Mexico*. Special Issue, *Reliability Engineering & System Safety*, 91(10-11):1105–1474, 2006.

[10] J. C. Helton, J. D. Johnson, C. J. Sallaberry, C. B. Storlie. Survey of sampling-based methods for uncertainty and sensitivity analysis. *Reliability Engineering & System Safety*, 91:1175-1209, 2006.

[11] R. L. Iman, W. J. Conover. The use of the rank transformation in regression. *Technometrics*, 21:499-509, 1979.

[12] R. L. Iman, W. J. Conover. A distribution-free approach to inducing rank correlation among input variables. *Communications in Statistics - Simulation and Computation*, 11:311–334, 1982.

[13] R. L. Iman, M. J. Shortencarier, J. D. Johnson. A FORTRAN 77 program and user's guide for the calculation of partial correlation and standardized regression coefficients. *SANDIA Report* SAND85-0044, Sandia National Laboratories, Albuquerque, 1985.

[14] G. Klir, M. J. Wiermann. *Uncertainty-Based Information. Elements of Generalized Information Theory*. Physica-Verlag, Heidelberg, 1998.

[15] W. T. Koiter. Current trends in the theory of buckling. In: B. Budiansky, editor. *Buckling of Structures. Proceedings IUTAM Symposium, Cambridge 1974*. Springer-Verlag, Berlin 1976.

[16] V. Kreinovich, S. A. Ferson. A new Cauchy-based black-box technique for uncertainty in risk analysis. *Reliability Engineering & and System Safety*, 85:267–279, 2004.

[17] M. D. McKay, R. J. Beckman, W. J. Conover. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21:239-245, 1979.

[18] I. Molchanov. *Theory of Random Sets*. Springer-Verlag, Berlin, 2005.

[19] H. T. Nguyen. *An Introduction to Random Sets*. Chapman & Hall, Boca Raton, 2006.

[20] M. Oberguggenberger, W. Fellin. Assessing the sensitivity of failure probabilities: a random set approach. In: G. Augusti, G. I. Schuëller, M. Ciampoli, editors. *Safety and Reliability of Engineering Systems and Structures. ICOSSAR 2005 - Rome*. Millpress, Rotterdam 2005, 1755–1760.

[21] M. Oberguggenberger, W. Fellin. Reliability bounds through random sets: nonparametric methods and geotechnical applications. *Computers & Structures*, to appear.

[22] A. Saltelli, I. M. Sobol. About the use of rank transformation in sensitivity analysis of model output. *Reliability Engineering & System Safety*, 50:225-239, 1995.

[23] J. Shao, D.-S. Tu. *The Jackknife and Bootstrap*. Springer Verlag, New York, 1995.

[24] G. I. Schuëller. On the treatment of uncertainties in structural mechanics and analysis. *Computers & Structures*, 85:235–243, 2007.

[25] I. M. Sobol. Sensitivity analysis for nonlinear mathematical models. *Mathematical Modeling & Computational Experiment*, 1:407-414, 1993.

[26] M. Stein. Large sample properies of simulations using Latin hypercube sampling. *Technometrics*, 29:143-151, 1987.

[27] F. Tonon, C. L. Pettit. Toward a definition and understanding of correlation for variables constrained by random relations. In: G. Augusti, G. I. Schuëller, M. Ciampoli, editors. *Safety and Reliability of Engineering Systems and Structures. ICOSSAR 2005 - Rome*. Millpress, Rotterdam 2005, 1741–1745.