

Relating practical representations of imprecise probabilities

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Abstract

There exist many practical representations of probability families that make them easier to handle. Among them are random sets, possibility distributions, probability intervals, Ferson's p-boxes and Neumaier's clouds. Both for theoretical and practical considerations, it is important to know whether one representation has the same expressive power than other ones, or can be approximated by other ones. In this paper, we mainly study the relationships between the two latter representations and the three other ones.

Keywords. Random Sets, possibility distributions, probability intervals, p-boxes, clouds.

1 Introduction

There are many representations of uncertainty. The theory of imprecise probabilities (including lower/upper previsions) [27] is the most general framework. It formally encompasses all the representations proposed by other uncertainty theories, regardless of their possible different interpretations.

The more general the theory, the more expressive it can be, and, usually, the more expensive it is from a computational standpoint. Simpler (but less flexible) representations can be useful if judged sufficiently expressive. They are mathematically and computationally easier to handle, and using them can greatly increase efficiency in applications.

Among these simpler representations are random sets [7], possibility distributions [28], probability intervals [2], p-boxes [15] and, more recently, clouds [21, 22]. With such a diversity of simplified representations, it is then natural to compare them from the standpoint of their expressive power. Building formal links between such representations also facilitates a unified handling of uncertainty, especially in propagation techniques exploiting uncertain data modeled by means of such representations. This is the pur-

pose of the present study. It extends some results by Baudrit and Dubois [1] concerning the relationships between p-boxes and possibility measures.

The paper is structured as follows: the first section briefly recalls the formalism of random sets, possibility distributions and probability intervals, as well as some existing results. Section 3 then focuses on p-boxes, first generalizing the notion of p-boxes to arbitrary finite spaces before studying the relationships of these generalized p-boxes with the three former representations. Finally, section 4 studies the relationships between clouds and the preceding representations. For the reader convenience, longer proofs are put in the appendix.

2 Preliminaries

In this paper, we consider that uncertainty is modeled by a family \mathcal{P} of probability assignments, defined over a finite referential $X = \{x_1, \dots, x_n\}$. We also restrict ourselves to families that can be represented by their lower and upper probability bounds, defined as follows:

$$\underline{P}(A) = \inf_{P \in \mathcal{P}} P(A) \text{ and } \overline{P}(A) = \sup_{P \in \mathcal{P}} P(A)$$

Let $\mathcal{P}_{\underline{P}, \overline{P}} = \{P | \forall A \subseteq X, \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}$. In general, we have $\mathcal{P} \subset \mathcal{P}_{\underline{P}, \overline{P}}$, since $\mathcal{P}_{\underline{P}, \overline{P}}$ can be seen as a projection of \mathcal{P} on events. Although they are already restrictions from more general cases, dealing with families $\mathcal{P}_{\underline{P}, \overline{P}}$ often remains difficult.

2.1 Random Sets

Formally, a random set [20] is a mapping Γ from a probability space to the power set $\wp(X)$ of another space X , also called a multi-valued mapping. This mapping induces lower and upper probabilities on X [7]. In the continuous case, the probability space is often $[0, 1]$ equipped with Lebesgue measure, and Γ is a point-to-interval mapping.

In the finite case, these lower and upper probabilities are respectively called belief and plausibility measures, and it can be shown that the belief measure is a ∞ -monotone capacity [4]. An alternative (and useful) representation of the random set consists of a normalized assignment of positive masses m over the power set $\wp(X)$ s.t. $\sum_{E \subseteq X} m(E) = 1$ and $m(\emptyset) = 0$ [25]. A set E that receives strictly positive mass is said to be focal. Belief and plausibility functions are then defined as follows:

$$\begin{aligned} Bel(A) &= \sum_{E, E \subseteq A} m(E) \\ Pl(A) &= 1 - Bel(A^c) = \sum_{E, E \cap A \neq \emptyset} m(E). \end{aligned}$$

The set

$$\mathcal{P}_{Bel} = \{P | \forall A \subseteq X, Bel(A) \leq P(A) \leq Pl(A)\}$$

is the probability family induced by the belief measure.

Although $2^{|X|}$ values are still needed to fully specify a general random set, the fact that they can be seen as probability assignments over subsets of X allows for simulation by means of some sampling process.

2.2 Possibility distributions

A possibility distribution π [12] is a mapping from X to the unit interval such that $\pi(x) = 1$ for some $x \in X$. Formally, a possibility distribution is the membership function of a fuzzy set. Several set-functions can be defined from a distribution π [11]:

- $\Pi(A) = \sup_{x \in A} \pi(x)$ (possibility measures);
- $N(A) = 1 - \Pi(A^c)$ (necessity measures);
- $\Delta(A) = \inf_{x \in A} \pi(x)$ (sufficiency measures).

Possibility degrees express the extent to which an event is plausible, i.e., consistent with a possible state of the world, necessity degrees express the certainty of events and sufficiency (also called guaranteed possibility) measures express the extent to which all states of the world where A occurs are plausible. They apply to so-called guaranteed possibility distributions [11] generally denoted by δ .

A possibility degree can be viewed as an upper bound of a probability degree [13]. Let

$$\mathcal{P}_\pi = \{P, \forall A \subseteq X, N(A) \leq P(A) \leq \Pi(A)\}$$

be the set of probability measures encoded by a possibility distribution π . A possibility distribution is also equivalent to a random set whose realizations are nested.

From a practical standpoint, possibility distributions are the simplest representation of imprecise probabilities (as for precise probabilities, only $|X|$ values are needed to specify them). Another important point is their interpretation in term of collection of confidence intervals [10], which facilitates their elicitation and makes them natural candidate for vague probability assessments (see [5]).

2.3 Probability intervals

Probability intervals are defined as lower and upper probability bounds restricted to singletons x_i . They can be seen as a collection of intervals $L = \{[l_i, u_i], i = 1, \dots, n\}$ defining a probability family:

$$\mathcal{P}_L = \{P | l_i \leq p(x_i) \leq u_i \forall x_i \in X\}.$$

Such families have been extensively studied in [2] by De Campos et al.

In this paper, we consider non-empty families (i.e. $\mathcal{P}_L \neq \emptyset$) that are reachable (i.e. each lower or upper bound on singletons can be reached by at least one probability assignment of the family \mathcal{P}_L). Conditions of non-emptiness and reachability respectively correspond to avoiding sure loss and achieving coherence in Walley's behavioural theory.

Given intervals L , lower and upper probabilities $\underline{P}(A), \overline{P}(A)$ are calculated by the following expressions

$$\begin{aligned} \underline{P}(A) &= \max(\sum_{x_i \in A} l_i, 1 - \sum_{x_i \notin A} u_i) \\ \overline{P}(A) &= \min(\sum_{x_i \in A} u_i, 1 - \sum_{x_i \notin A} l_i) \end{aligned} \quad (1)$$

De Campos et al. have shown that these bounds are Choquet capacities of order 2 (\underline{P} is a convex capacity).

The problem of approximating \mathcal{P}_L by a random set has been treated in [17] and [8]. While in [17], Lemmer and Kyburg find a random set m_1 that is an inner approximation of \mathcal{P}_L s.t. $Bel_1(x_i) = l_i$ and $Pl_1(x_i) = u_i$, Denoeux [8] extensively studies methods to build a random set that is an outer approximation of \mathcal{P}_L . The problem of finding a possibility distribution approximating \mathcal{P}_L is treated by Masson and Denoeux in [19].

Two common cases where probability intervals can be encountered as models of uncertainty are confidence intervals on parameters of multinomial distributions built from sample data, and expert opinions providing such intervals.

3 P-boxes

We first recall some usual notions on the real line that will be generalized in the sequel.

Let \Pr be a probability function on the real line with density p . The *cumulative* distribution of \Pr is denoted F^p and is defined by $F^p(x) = \Pr((-\infty, x])$.

Let $F_1(x)$ and $F_2(x)$ be two *cumulative* distributions. Then, $F_1(x)$ is said to stochastically dominate $F_2(x)$ iff $F_1(x) \leq F_2(x) \forall x$.

A P-box [15] is defined by a pair of cumulative distributions $\underline{F} \leq \overline{F}$ (\underline{F} stochastically dominates \overline{F}) on the real line. It brackets the cumulative distribution of an imprecisely known probability function with density p s.t. $\underline{F}(x) \leq F^p(x) \leq \overline{F}(x) \quad \forall x \in \mathfrak{R}$.

3.1 Generalized Cumulative Distributions

Interestingly, the notion of cumulative distribution is based on the existence of the natural ordering of numbers. Consider a probability assignment (probability vector) $\lambda = (\lambda_1 \dots \lambda_n)$ defined over the finite space X ; λ_i denotes the probability $\Pr(x_i)$ of the i -th element x_i , and $\sum_{j=1}^n \lambda_j = 1$. In this case, no natural notion of cumulative distribution exists. In order to make sense of this notion over X , one must equip it with a complete preordering \leq_R , which is a reflexive, complete and transitive relation. An R -downset is of the form $\{x_i : x_i \leq_R x\}$, and denoted $(x]_R$.

Definition 1. [9] *The generalized R -cumulative distribution of a probability assignment λ on a finite, completely preordered set (X, \leq_R) is the function $F_R^\lambda : X \rightarrow [0, 1]$ defined by $F_R^\lambda(x) = \Pr((x]_R)$.*

The usual notion of stochastic dominance can also be defined for generalized cumulative distributions. Consider another probability assignment $\kappa = (\kappa_1 \dots \kappa_n)$ on X . The corresponding R -dominance relation of λ over κ can be defined by the pointwise inequality $F_R^\lambda < F_R^\kappa$. Clearly, a generalized cumulative distribution can always be considered as a simple one, up to a reordering of elements.

Any generalized cumulative distribution F_R^λ with respect to a complete preorder \leq_R on X , of a probability measure \Pr , with assignment λ on X , can also be used as a possibility distribution π_R whose associated measure dominates \Pr , i.e. $\max_{x \in A} F_R^\lambda(x) \geq \Pr(A), \forall A \subseteq X$. This is because a (generalized) cumulative distribution is constructed by computing the probabilities of events $\Pr(A)$ in a nested sequence of downsets $(x_i]_R$. [10].

3.2 Generalized p-box

Using the generalizations of the notions of cumulative distributions and of stochastic dominance described in section 3.1, we define a generalized p-box as follows

Definition 2. *A R -P-box on a finite, completely preordered set (X, \leq_R) is a pair of R -cumulative distributions $F_R^\lambda(x)$ and $F_R^\kappa(x)$, s.t. $F_R^\lambda(x) \leq F_R^\kappa(x)$ (i.e. κ is a probability assignment R -dominated by λ)*

The probability family induced by a R -P-box is

$$\mathcal{P}_{p\text{-box}} = \{P | \forall x, F_R^\lambda(x) \leq F_R(x) \leq F_R^\kappa(x)\}$$

If we choose a relation R with $x_i \leq_R x_j$ iff $i < j$, and, $\forall x_i \in X$, consider the sets $A_i = (x_i]_R$, it comes down to a family of nested confidence sets $\emptyset \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subset X$. The family $\mathcal{P}_{p\text{-box}}$ can then be represented by the following restrictions on probability measures [9]:

$$\alpha_i \leq P(A_i) \leq \beta_i \quad i = 1, \dots, n \quad (2)$$

with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n \leq 1$ and $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n \leq 1$. Choosing $X = \mathfrak{R}$ and $A_i = (-\infty, x_i]$, it is easy to see that we find back the usual definition of P-boxes.

A generalized cumulative distribution being fully specified by $|X|$ values, it follows that $2|X|$ values must be given to completely determine a generalized p-box. Moreover, we can interpret p-boxes as a collection of nested confidence intervals with upper and lower probability bounds (which could come, for example, from expert elicitation). In order to make notation simpler, the upper and lower cumulative distributions will respectively be noted F^* , F_* in the sequel and, unless stated otherwise, we will consider (without loss of generality) the order R s.t. $x_i \leq_R x_j$ iff $i < j$ with the associated nested sets A_i .

3.3 Generalized P-boxes in the setting of possibility theory

Given that sets A_i can be interpreted as nested confidence intervals with upper and lower bounds, it is natural to search a connection with possibility theory, since possibility distributions can be interpreted as a collection of nested confidence intervals (a natural way of expressing expert knowledge). We thus have the following proposition

Proposition 1. *A family $\mathcal{P}_{p\text{-box}}$ described by a generalized P-box can be encoded by a pair of possibility distributions π_1, π_2 s.t. $\mathcal{P}_{p\text{-box}} = \mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$ with $\pi_1(x) = F^*(x)$ and $\pi_2(x) = 1 - F_*(x)$*

Proof of proposition 1. Consider the definition of a generalized p-box and the fact that a generalized cumulative distribution can be used as a possibility distribution π_R dominating the probability distribution \Pr (see section 3.1). Then, the set of constraints $(P(A_i) \geq \alpha_i)_{i=1,n}$ from equation (2) generates a possibility distribution π_1 and the set of constraints $(P(A_i^c) \geq 1 - \beta_i)_{i=1,n}$ generates a possibility distribution π_2 . Clearly $\mathcal{P}_{p\text{-box}} = \mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$. \square

3.4 Generalized P-boxes are special case of random sets

The following proposition was proved in [9]

Proposition 2. A family $\mathcal{P}_{p\text{-box}}$ described by a generalized P-box can be encoded by a random set s.t. $\mathcal{P}_{p\text{-box}} = \mathcal{P}_{Bel}$.

Algorithm 1: R-P-box \rightarrow random set

Input: Nested sets $\emptyset, A_1, \dots, A_n, X$ and bounds

$$\alpha_i, \beta_i$$

Output: Equivalent random set

for $k = 1, \dots, n + 1$ **do**

\lfloor Build partition $F_i = A_i \setminus A_{i-1}$

Rank α_i, β_i increasingly

for $k = 0, \dots, 2n + 1$ **do**

 Rename α_i, β_i by γ_l s.t.

$$\alpha_0 = \gamma_0 = 0 \leq \gamma_1 \leq \dots \leq \gamma_l \leq \dots \leq \gamma_{2n} \leq 1 =$$

$$\lfloor \gamma_{2n+1} = \beta_{n+1}$$

Define focal set $E_0 = \emptyset$

for $k = 1, \dots, 2n + 1$ **do**

if $\gamma_{k-1} = \alpha_i$ **then**

$\lfloor E_k = E_{k-1} \cup F_{i+1}$

if $\gamma_{k-1} = \beta_i$ **then**

$\lfloor E_k = E_{k-1} \setminus F_i$

 Set $m(E_k) = \gamma_k - \gamma_{k-1}$

Algorithm 1 provides an easy way to build the random set encoding a generalized p-box. It is similar to algorithms given in [16, 24], and extends them to more general spaces. The main idea of the algorithm is to use the fact that a generalized p-box can be seen as a random set whose focal elements are unions of adjacent sets in a partition. Thanks to the nested nature of sets A_i , we can build a partition of X made of $F_i = A_i \setminus A_{i-1}$, and then add or subtract consecutive elements of this partition to build the focal sets (of the form $\bigcup_{j \leq i \leq k} F_i$) of the random set equivalent to the generalized p-box. The following example illustrates both the notion of generalized p-box and algorithm 1.

Example 1. Consider a space X made of six elements $\{x_1, \dots, x_6\}$ (These elements could be, for instance, successive components on a production line). For various reasons, one can only observe whether

the event $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_2, x_3\}$, $A_3 = \{x_1, x_2, x_3, x_4, x_5\}$ or the whole X happens. Suppose an expert must evaluate the likelihood of these events, and only gives us probability intervals :

$$P(A_1) \in [0, 0.3] \quad P(A_2) \in [0.2, 0.7] \quad P(A_3) \in [0.5, 0.9]$$

So we have a generalized p-box, the order of the elements being determined by the possible observations (notice that we are indifferent to the order of x_1, x_2 and of x_4, x_5). Applying algorithm 1, we have :

$$F_1 = \{x_1, x_2\} \quad F_2 = \{x_3\} \quad F_3 = \{x_4, x_5\} \quad F_4 = \{x_6\}$$

and

$$0(\alpha_0) \leq 0(\alpha_1) \leq 0.2(\alpha_2) \leq 0.3(\beta_1) \leq 0.5(\alpha_3) \\ \leq 0.7(\beta_2) \leq 0.9(\beta_3) \leq 1$$

which gives us the following corresponding random set

$$m(E_1) = m(\{x_1, x_2\}) = 0 \quad m(E_2) = m(\{x_1, x_2, x_3\}) = 0.2$$

$$m(E_3) = m(\{x_1, x_2, x_3, x_4, x_5\}) = 0.1 \quad m(E_4) = m(\{x_3, x_4, x_5\}) = 0.2$$

$$m(E_5) = m(\{x_3, x_4, x_5, x_6\}) = 0.2 \quad m(E_6) = m(\{x_4, x_5, x_6\}) = 0.2$$

$$m(E_7) = m(\{x_6\}) = 0.1$$

which makes the imprecision of the available information more visible.

3.5 Generalized P-boxes and probability intervals

Provided an order R has been defined on elements x_i , a method to build a p-box from probability intervals L can be easily derived from equations (1). Lower and upper generalized cumulative distributions can be computed as follows

$$F_*(x_i) = \underline{P}(A_i) = \max\left(\sum_{x_i \in A_i} l_j, 1 - \sum_{x_i \notin A_i} u_j\right)$$

$$F^*(x_i) = \overline{P}(A_i) = \min\left(\sum_{x_i \in A_i} u_i, 1 - \sum_{x_i \notin A_i} l_i\right) \quad (3)$$

Transforming a p-box into probability intervals is also an easy task. First, let us assume that each element F_i of the partition used in algorithm 1 is reduced to a singleton x_i . Corresponding probability intervals are then given by the two following formulas:

$$\underline{P}(F_i) = \underline{P}(x_i) = l_i = \max(0, \alpha_i - \beta_{i-1})$$

$$\overline{P}(F_i) = \overline{P}(x_i) = u_i = \beta_i - \alpha_{i-1}$$

if a set F_i is made of n elements x_{i1}, \dots, x_{in} , it is easy to see that $l(x_{ij}) = 0$ and that $u(x_{ij}) = \overline{P}(F_i)$, since $x_{ij} \in F_i$.

Let us note that transforming probability intervals into p-boxes (and conversely) generally loses information, except in the degenerated cases of precise probability assignment and of total ignorance. If no obvious

order relation R between elements x_i is to be privileged, and if one wants to transform probability intervals into generalized p-boxes, we think that a good choice for the order R is the one s.t.

$$\sum_{i=1}^n F^*(x_i) - F_*(x_i)$$

is minimized, so that a minimal amount of information is lost in the process.

Another interesting fact to pinpoint is that both cumulative distributions given by equations (3) can be interpreted as possibility distributions dominating the family \mathcal{P}_L (for F_* , the associated possibility distribution is $1 - F_*$). Thus, computing either F^* or F_* is a method to find a possibility distribution approximating \mathcal{P}_L , which is different from the one proposed by Masson and Denoeux [19].

4 Clouds

We begin this section by recalling basic definitions and results due to Neumaier [21], cast in the terminology of fuzzy sets and possibility theory. A *cloud* is an Interval-Valued Fuzzy Set F such that $(0, 1) \subseteq \cup_{x \in X} F(x) \subseteq [0, 1]$, where $F(x)$ is an interval $[\delta(x), \pi(x)]$. In the following, it is either defined on a finite space X , or it is a continuous interval-valued fuzzy interval (IVFI) on the real line (a “cloudy” interval). In the latter case each fuzzy set has cuts that are closed intervals. When the upper membership function coincides with the lower one, ($\delta = \pi$) the cloud is called *thin*, and when the lower membership function is identically 0, the cloud is called *fuzzy* by Neumaier. Let us note that these names are somewhat counter-intuitive, since a *thin* cloud correspond to a fuzzy set with precise membership function, while a *fuzzy* cloud is equivalent to a probability family modeled by a possibility distribution.

A random variable x with values in X is said to belong to a cloud F if and only if $\forall \alpha \in [0, 1]$:

$$P(\delta(x) \geq \alpha) \leq 1 - \alpha \leq P(\pi(x) > \alpha) \quad (4)$$

under all suitable measurability assumptions.

If X is a finite space of cardinality n , a *cloud* can be defined by the following restrictions :

$$P(B_i) \leq 1 - \alpha_i \leq P(A_i) \text{ and } B_i \subseteq A_i, \quad (5)$$

where $1 = \alpha_0 > \alpha_1 > \alpha_2 > \dots > \alpha_n > \alpha_{n+1} = 0$ and $\emptyset = A_0 \subset A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} = X$; $\emptyset = B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots \subseteq B_n \subseteq B_{n+1} = X$. The confidence sets A_i and B_i are respectively the strong and regular α -cut of fuzzy sets π and δ ($A_i = \{x_i, \pi(x_i) > \alpha_{i+1}\}$ and $B_i = \{x_i, \delta(x_i) \geq \alpha_{i+1}\}$).

As for probability intervals and p-boxes, eliciting a cloud requires $2|X|$ values.

4.1 Clouds in the setting of possibility theory

Let us first recall the following result regarding possibility measures (see [10]):

Proposition 3. $P \in \mathcal{P}_\pi$ if and only if $1 - \alpha \leq P(\pi(x) > \alpha), \forall \alpha \in (0, 1]$

The following proposition directly follows

Proposition 4. A probability family $\mathcal{P}_{\delta, \pi}$ described by the cloud (δ, π) is equivalent to the family $\mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$ described by the two possibility distributions π and $1 - \delta$.

Proof of proposition 4. Consider a cloud (δ, π) , and define $\bar{\pi} = 1 - \delta$. Note that $P(\delta(x) \geq \alpha) \leq 1 - \alpha$ is equivalent to $P(\bar{\pi} > \beta) \geq 1 - \beta$, letting $\beta = 1 - \alpha$. So it is clear from equation (4) that probability measure P is in the cloud (δ, π) if and only if it is in $\mathcal{P}_\pi \cap \mathcal{P}_{\bar{\pi}}$. So a cloud is a family of probabilities dominated by two possibility distributions (see [14]). \square

This property is common to generalized p-boxes and clouds: they define probability families upper bounded by two possibility measures. It is then natural to investigate their relationships.

4.2 Finding clouds that are generalized p-boxes

Proposition 5. A cloud is a generalized p-box iff $\{A_i, B_i, i = 1, \dots, n\}$ form a nested sequence of sets (i.e. there is a linear preordering with respect to inclusion)

Proof of proposition 5. Assume the sets A_i and B_j form a globally nested sequence whose current element is C_k . Then the set of constraints defining a cloud can be rewritten in the form $\gamma_k \leq P(C_k) \leq \beta_k$, where $\gamma_k = 1 - \alpha_i$ and $\beta_k = \min\{1 - \alpha_j : A_i \subseteq B_j\}$ if $C_k = A_i$; $\beta_k = 1 - \alpha_i$ and $\gamma_k = \max\{1 - \alpha_j : A_j \subseteq B_i\}$ if $C_k = B_i$.

Since $1 = \alpha_0 > \alpha_1 > \dots > \alpha_n < \alpha_{n+1} = 0$, these constraints are equivalent to those of a generalized p-box. But if $\exists B_j, A_i$ with $j > i$ s.t. $B_j \not\subseteq A_i$ and $A_i \not\subseteq B_j$, then the cloud is not equivalent to a p-box, since confidence sets would no more form a complete preordering with respect to inclusion. \square

In term of pairs of possibility distributions, it is now easy to see that a cloud (δ, π) is a generalized p-box

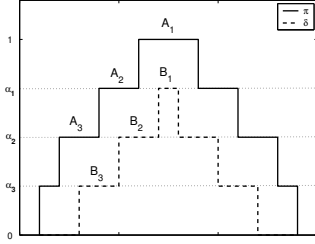


Figure 1: Comonotonic cloud

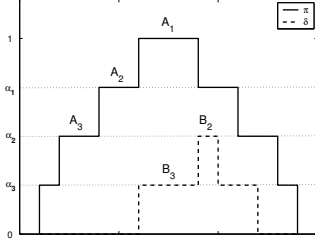


Figure 2: Non-Comonotonic cloud

if and only if π and δ are comonotonic. We will thus call such clouds comonotonic clouds. If a cloud is comonotonic, we can thus directly adapt the various results obtained for generalized p-boxes. In particular, because comonotonic clouds are generalized p-boxes, algorithm 1 can be used to get the corresponding random set. Notions of comonotonic and non-comonotonic clouds are respectively illustrated by figures 1 and 2

4.3 Characterizing and approximating non-comonotonic clouds

The following proposition characterizes probability families represented by most non-comonotonic clouds, showing that the distinction between comonotonic and non-comonotonic clouds makes sense (since the latter cannot be represented by random sets).

Proposition 6. *If (δ, π) is a non-comonotonic cloud for which there are two overlapping sets A_i, B_j that are not nested (i.e. $A_i \cap B_j \neq \{A_i, B_j, \emptyset\}$), then the lower probability of the induced family $\mathcal{P}_{\delta, \pi}$ is not even 2-monotone.*

The proof can be found in the appendix.

Remark 1. *The case for which we have $B_j \cap A_i \in \{A_i, B_j\}$ for all pairs A_i, B_j is the case of comonotonic clouds. Now, if a cloud is such that for all pairs $A_i, B_j : B_j \cap A_i \in \{A_i, B_j, \emptyset\}$ with at least one empty intersection, then it is still a random set, but no longer a generalized p-box. Let us note that this special case can only occur for discrete clouds.*

Since it can be computationally difficult to work with

capacities that are not 2-monotone, one could wish to work either with outer or inner approximations. We propose two such approximations, which are easy to compute and respectively correspond to necessity (possibility) measures and belief (plausibility) measures.

Proposition 7. *If $\mathcal{P}_{\delta, \pi}$ is the probability family described by the cloud (δ, π) on a referential X , then, the following bounds provide an outer approximation of the range of $P(A)$:*

$$\begin{aligned} \max(N_\pi(A), N_{1-\delta}(A)) &\leq P(A) \leq \\ \min(\Pi_\pi(A), \Pi_{1-\delta}(A)) &\forall A \subset X \end{aligned} \quad (6)$$

Proof of proposition 7. Since we have that $\mathcal{P}_{\delta, \pi} = \mathcal{P}_{1-\delta} \cap \mathcal{P}_\pi$, and given the bounds defined by each possibility distributions, it is clear that equation 6 give bounds of $P(A)$. \square

We can check that the bounds given by equation (6) are the one considered by Neumaier in [21]. Since these bounds are, in general, not the infimum and supremum of $P(A)$ on $\mathcal{P}_{\delta, \pi}$, Neumaier's claim that clouds are only vaguely related to Walley's previsions or random sets is not surprising. Nevertheless, if we consider the relationship between clouds and possibility distributions, taking this outer approximation, that is very easy to compute, seems very natural.

Nevertheless, these bounds are not, in general, the infimum and the supremum of $P(A)$ over $\mathcal{P}_{\delta, \pi}$. To see this, consider a discrete cloud made of four non-empty elements A_1, A_2, B_1, B_2 . It can be checked that

$$\begin{aligned} \pi(x) &= 1 \text{ if } x \in A_1; \\ &= \alpha_1 \text{ if } x \in A_2 \setminus A_1; \\ &= \alpha_2 \text{ if } x \notin A_2. \\ \delta(x) &= \alpha_1 \text{ if } x \in B_1; \\ &= \alpha_2 \text{ if } x \in B_2 \setminus B_1; \\ &= 0 \text{ if } x \notin B_2. \end{aligned}$$

Since $P(A_2) \geq 1 - \alpha_2$ and $P(B_1) \leq 1 - \alpha_1$, from (5), we can easily check that $\underline{P}(A_2 \setminus B_1) = \underline{P}(A_2 \cap B_1^c) = \alpha_1 - \alpha_2$. Now, $N_\pi(A_2 \cap B_1^c) = \min(N_\pi(A_2), N_\pi(B_1^c)) = 0$ since $\Pi_\pi(B_1) = 1$ and $B_1 \subseteq A_1$. Considering distribution δ , we can have $N_{1-\delta}(A_2 \cap B_1^c) = \min(N_{1-\delta}(A_2), N_{1-\delta}(B_1^c)) = 0$ since $N_{1-\delta}(A_2) = \Delta_\delta(A_2^c) = 0$ since $B_2 \subseteq A_2$. Equation (6) can thus result in a trivial lower bound, different from $\underline{P}(A_2 \setminus B_1)$.

The next proposition provides an inner approximation of $\mathcal{P}_{\delta, \pi}$

Proposition 8. *Given the sets $\{B_i, A_i, i = 1, \dots, n\}$ inducing the distributions (δ, π) of a cloud and the*

corresponding α_i , the belief and plausibility measures of the random set s.t. $m(A_i \setminus B_{i-1}) = \alpha_{i-1} - \alpha_i$ are inner approximations of $\mathcal{P}_{\delta,\pi}$.

It is easy to see that this random set can always be defined. We can see that it is always an inner approximation by using the contingency matrix advocated in the proof of proposition 6 (see appendix). In this matrix, the random set defined above comes down to concentrating weights on diagonal elements. This inner approximation is exact in case of comonotonicity or when we have $A_i \cap B_j \in \{A_i, B_j, \emptyset\}$ for any pair of sets A_i, B_j defining the clouds.

4.4 A note on thin and continuous clouds

Thin clouds ($\delta = \pi$) constitute an interesting special case of clouds. In this latter case, conditions defining clouds are reduced to

$$P(\pi(x) \geq \alpha) = P(\pi(x) > \alpha) = 1 - \alpha, \forall \alpha \in (0, 1).$$

On finite sets these constraints are generally contradictory, because $P(\pi(x) \geq \alpha) > P(\pi(x) > \alpha)$ for some α , hence the following theorem:

Proposition 9. *If X is finite, then $\mathcal{P}(\pi) \cap \mathcal{P}(1 - \pi)$ is empty.*

which is proved in [14], where it is also shown that this emptiness is due to finiteness. A simple shift of indices solves the difficulty. Let $\pi(u_i) = \alpha_i$ such that $\alpha_1 = 1 > \dots > \alpha_n > \alpha_{n+1} = 0$. Consider $\delta(u_i) = \alpha_{i+1} < \pi_1(u_i)$. Then $\mathcal{P}(\pi) \cap \mathcal{P}(1 - \delta)$ contains the unique probability measure P such that the probability weight attached to u_i is $p_i = \alpha_i - \alpha_{i+1}, \forall i = 1 \dots n$. To see it, refer to equation (5), and note that in this case $A_i = B_i$.

In the continuous case, a thin cloud is non-trivial. The inclusions $[\delta(x) \geq \alpha] \subseteq [\pi(x) > \alpha]$ (corresponding to $B_i \subseteq A_i$) again do not work but we may have $P(\pi(x) \geq \alpha) = P(\pi(x) > \alpha) = 1 - \alpha, \forall \alpha \in (0, 1)$. For instance, a cumulative distribution function, viewed as a tight p-box, defines a thin cloud containing the only random variable having this cumulative distribution (the "right" side of the cloud is rejected to ∞). In fact, it was suggested in [14] that a thin cloud contains in general an infinity of probability distributions.

Insofar as Proposition 5 can be extended to the reals (this could be shown, for instance, by proving the convergence of some finite outer and inner approximations of the continuous model, or by using the notion of directed set [5] to prove the complete monotonicity of the model), then a thin cloud can be viewed as a generalized p-box and is thus a (continuous) belief function with uniform mass density, whose focal

sets are doubletons of the form $\{x(\alpha), y(\alpha)\}$ where $\{x : \pi(x) \geq \alpha\} = [x(\alpha), y(\alpha)]$. It is defined by the Lebesgue measure on the unit interval and the multimaping $\alpha \rightarrow \{x(\alpha), y(\alpha)\}$. This result gives us a nice way to characterize the infinite set of random variables contained in a thin cloud. In particular, concentrating the mass density on elements $x(\alpha)$ or on elements $y(\alpha)$ would respectively give the upper and lower cumulative distributions that would have been associated to the possibility distribution π alone (let us note that every convex mixture of those two cumulative distributions would also be in the thin cloud). It is also clear that $Bel(\pi(x) \geq \alpha) = 1 - \alpha$. More generally, if Proposition 5 holds in the continuous case, a comonotonic cloud can be characterized by a continuous belief function [26] with uniform mass density, whose focal sets would be unions of disjoint intervals of the form $[x(\alpha), u(\alpha)] \cup [v(\alpha), y(\alpha)]$ where $\{x : \pi(x) \geq \alpha\} = [x(\alpha), y(\alpha)]$ and $\{x : \delta(x) \geq \alpha\} = [u(\alpha), v(\alpha)]$.

4.5 Clouds and probability intervals

Since probability intervals are 2-monotone capacities, while clouds are either ∞ -monotone capacities or not even 2-monotone capacities, there is no direct correspondence between probability intervals and clouds. Nevertheless, given previous results, we can easily build a cloud approximating a family \mathcal{P}_L defined by a set L of probability intervals (but perhaps not the most "specific" one): indeed, any generalized p-box built from the probability intervals is a comonotonic cloud encompassing the family \mathcal{P}_L .

Finding the "best" (i.e. keeping as much information as possible, given some information measure) method to transform probability intervals into cloud is an open problem. Any such transformation in the finite case should follow some basic requirements such as:

1. Since clouds can model precise probability assignments, the method should insure that a precise probability assignment will be transformed into the corresponding (almost thin) cloud.
2. Given a set L of probability intervals, the transformed cloud $[\delta, \pi]$ should contain \mathcal{P}_L (i.e. $\mathcal{P}_{\delta,\pi} \subset \mathcal{P}_L$) while being as close to it as possible.

Let us note that using the transformation proposed in section 3.5 for generalized p-boxes satisfies these two requirements. Another solution is to extend Mason and Denoeux's [19] method that builds a possibility distribution covering a set of probability intervals, completing it by a lower distribution δ (due to lack of space, we do not explore this alternative here).

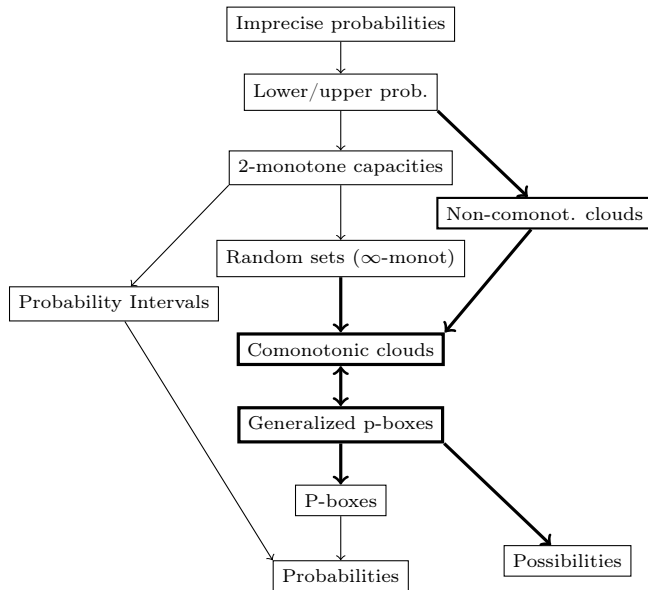


Figure 3: Representations relationships. $A \longrightarrow B$: B is a special case of A

5 Conclusions

Figure 3 summarizes our results cast in a more general framework of imprecise probability representations (our main contributions in boldface).

In this paper, we have considered many practical representations of imprecise probabilities, which are easier to handle than general probability families. They often require less data to be fully specified and they allow many mathematical simplifications, which may prove to increase computational efficiency (except, perhaps, for non-comonotonic clouds).

Some clarifications are provided concerning the situation of the cloud formalism. The fact that non-comonotonic clouds are not even 2-monotone capacities tends to indicate that, from a computational standpoint, they may be more difficult to exploit than the other formalisms. Nevertheless, as far as we know, they are the only simple model generating capacities that are not 2-monotone.

A work that remains to be done to a large extent is to evaluate the validity and the usefulness of these representations, particularly from a psychological standpoint (even if some of it has already been done [23, 18]). Another issue is to extend presented results to continuous spaces or to general lower/upper previsions (by using results from, for example [26, 6]). Finally, a natural continuation to this work is to explore various aspects of each formalisms in a manner similar to the one of De campos et al. [2]. What be-

comes of random sets, possibility distributions, generalized p-boxes and clouds after fusion, marginalization, conditioning or propagation? Do they preserve the representation? and under which assumptions? To what extent are these representations informative? Can they easily be elicited or integrated? If many results already exist for random sets and possibility distributions, there are fewer results for generalized p-boxes or clouds, due to their novelty.

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Appendix

Proof of proposition 6 (sketch). Our proof uses the following result by Chateauneuf [3]: Let m_1, m_2 be two random sets with focal sets $\mathcal{F}_1, \mathcal{F}_2$, each of them respectively defining a probability family $\mathcal{P}_{Bel_1}, \mathcal{P}_{Bel_2}$. Here, we assume that those families are "compatible" (i.e. $\mathcal{P}_{Bel_1} \cap \mathcal{P}_{Bel_2} \neq \emptyset$).

Then, the result from Chateauneuf states the following : the lower probability $\underline{P}(E)$ of the event E on $\mathcal{P}_{Bel_1} \cap \mathcal{P}_{Bel_2}$ is equal to the least belief measure $Bel(E)$ that can be computed on the set of joint normalized random sets with marginals m_1, m_2 . More formally, let us consider a set \mathcal{Q} s.t. $Q \in \mathcal{Q}$ iff

- $Q(A, B) > 0 \Rightarrow A \times B \in \mathcal{F}_1 \times \mathcal{F}_2$ (masses over the cartesian product of focal sets)
- $A \cap B = \emptyset \Rightarrow Q(A, B) = 0$ (normalization constraints)
- $m_1(A) = \sum_{B \in \mathcal{F}_2} Q(A, B)$ and $m_2(B) = \sum_{A \in \mathcal{F}_1} Q(A, B)$ (marginal constraints)

and the lower probability $\underline{P}(E)$ is given by the following equation

$$\underline{P}(E) = \min_{Q \in \mathcal{Q}} \sum_{(A \cap B) \subseteq E} Q(A, B) \quad (7)$$

where \mathcal{Q} is the set of joint normalized random sets. This result can be applied to clouds, since the family described by a cloud is the intersection of two families modeled by possibility distributions.

To illustrate the general proof, we will restrict ourselves to a 4-set cloud (the most simple non-trivial cloud that can be found). We thus consider four sets A_1, A_2, B_1, B_2 s.t. $A_1 \subset A_2, B_1 \subset B_2, B_i \subset A_i$ together with two values α_1, α_2 s.t. $1(=\alpha_0) > \alpha_1 > \alpha_2 > 0(=\alpha_3)$ and the cloud is defined by enforcing the inequalities $P(B_i) \leq 1 - \alpha_i \leq P(A_i)$ $i = 1, 2$. The random sets equivalent to the possibility distributions $\pi, 1 - \delta$ are summarized in the following table:

π	$1 - \delta$
$m(A_1) = 1 - \alpha_1$	$m(B_0^c = X) = 1 - \alpha_1$
$m(A_2) = \alpha_1 - \alpha_2$	$m(B_1^c) = \alpha_1 - \alpha_2$
$m(A_3 = X) = \alpha_2$	$m(B_2^c) = \alpha_2$

Furthermore, we add the constraint $A_1 \cap B_2 \neq \{A_1, B_2, \emptyset\}$, related to the non-monotonicity of the cloud. We then have the following contingency matrix, where the mass m_{ij} is assigned to the intersection of the corresponding sets at the beginning of line i and the top of column j :

	$B_0^c = X$	B_1^c	B_2^c	\sum
A_1	m_{11}	m_{12}	m_{13}	$1 - \alpha_1$
A_2	m_{21}	m_{22}	m_{23}	$\alpha_1 - \alpha_2$
$A_3 = X$	m_{31}	m_{32}	m_{33}	α_2
\sum	$1 - \alpha_1$	$\alpha_1 - \alpha_2$	α_2	1

We now consider the four events $A_1, B_2^c, A_1 \cap B_2^c, A_1 \cup B_2^c$. Given the above contingency matrix, we immediately have $\underline{P}(A_1) = 1 - \alpha_1$ and $\underline{P}(B_2^c) = \alpha_2$, since A_1 only includes the (joint) focal sets in the first line and B_2^c in the third column.

It is also easy to see that $\underline{P}(A_1 \cap B_2^c) = 0$, by considering the mass assignment $m_{ii} = \alpha_{i-1} - \alpha_i$ (we then have $m_{13} = 0$, which is the mass of the only joint focal set included in $A_1 \cap B_2^c$).

Now, concerning $\underline{P}(A_1 \cup B_2^c)$, let us consider the following mass assignment:

$$\begin{aligned} A_2 \cap B_1^c &: m_{22} = \alpha_1 - \alpha_2 \\ A_3 \cap B_0^c &: m_{31} = \min(1 - \alpha_1, \alpha_2) \\ A_1 \cap B_0^c &: m_{11} = 1 - \alpha_1 - m_{31} \\ A_3 \cap B_2^c &: m_{33} = \alpha_2 - m_{31} \\ A_1 \cap B_2^c &: m_{13} = m_{31} \end{aligned}$$

it can be checked that this mass assignment satisfies the constraints of the contingency matrix, and that the only joint focal sets included in $A_1 \cup B_2^c$ are those with masses m_{11}, m_{33}, m_{13} . Summing these masses, we have $\underline{P}(A_1 \cup B_2^c) = \max(\alpha_2, 1 - \alpha_1)$. Hence:

$$\begin{aligned} \underline{P}(A_1 \cup B_2^c) + \underline{P}(A_1 \cap B_2^c) &< \underline{P}(B_2^c) + \underline{P}(A_1) \\ \max(\alpha_2, 1 - \alpha_1) &< 1 - \alpha_1 + \alpha_2 \end{aligned}$$

an inequality that clearly violates the 2-monotonicity property. We have thus shown that in the 4-set case, 2-monotonicity never holds for families modeled by non-comonotonic clouds.

Now, in the general case, we have the following contingency matrix

	B_0^c	\cdot	B_j^c	\cdot	B_n^c	\sum
A_1	m_{11}					$1 - \alpha_1$
\cdot						
A_i			$m_{i(j+1)}$			$\alpha_{i-1} - \alpha_i$
\cdot						
A_{n+1}					$m_{(n+1)(n+1)}$	α_n
\sum	$1 - \alpha_1$		$\alpha_j - \alpha_{j+1}$		α_n	

Under the hypothesis of proposition 6, there are two sets A_i, B_j s.t. $A_i \cap B_j \neq \{A_i, B_j, \emptyset\}$. Due to the inclusion relationships between the sets, and similarly to what was done in the 4-set case, we have

$$\begin{aligned} \underline{P}(A_i) &= 1 - \alpha_i \\ \underline{P}(B_j^c) &= \alpha_j \\ \underline{P}(A_i \cap B_j^c) &= 0 \end{aligned}$$

Next, let us concentrate on event $A_i \cup B_j^c$ (which is different from X by hypothesis). Let us suppose that $m_{kk} = \alpha_{k-1} - \alpha_k$, except for masses $m_{(j+1)i}, m_{ii}, m_{i(j+1)}, m_{(j+1)(j+1)}$. This is similar to the 4-set case with masses $m_{(j+1)i}, m_{ii}, m_{i(j+1)}, m_{(j+1)(j+1)}$ and we define the following assignment

$$\begin{aligned} A_i \cap B_j^c &: m_{i(j+1)} = \min(\alpha_{i-1} - \alpha_i, \alpha_j - \alpha_{j+1}) \\ A_i \cap B_{i-1}^c &: m_{ii} = \alpha_{i-1} - \alpha_i - m_{i(j+1)} \\ A_{j+1} \cap B_j^c &: m_{(j+1)(j+1)} = \alpha_j - \alpha_{j+1} - m_{i(j+1)} \\ A_{j+1} \cap B_{i-1}^c &: m_{(j+1)i} = \min(\alpha_{i-1} - \alpha_i, \alpha_j - \alpha_{j+1}) \end{aligned}$$

Given this specific mass assignment (which is always inside the set \mathcal{Q}), and summing assignments given to subsets of $A_i \cup B_j^c$, the following inequality results:

$$\underline{P}(A_i \cup B_j^c) \leq \max(\alpha_{j+1} + 1 - \alpha_i, \alpha_j + 1 - \alpha_{i-1})$$

so,

$$\underline{P}(A_i \cup B_j^c) + \underline{P}(A_i \cap B_j^c) < \underline{P}(A_i) + \underline{P}(B_j^c),$$

which clearly violates the 2-monotonicity property. \square