Compositional Models of Belief Functions

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Abstract

After it has been successfully done in probability and possibility theories, the paper is the first attempt to introduce the operator of composition also for belief functions. We prove that the proposed definition preserves all the necessary properties of the operator enabling us to define compositional models as an efficient tool for multidimensional models representation.

Keywords. Belief function, basic assignment, multidimensional frame of discernment, operator of composition, perfect sequence.

1 Introduction

Last years of the last century witnessed emergence of a new approach to efficient representation of multidimensional probability distributions. This approach, which is an alternative to Graphical Markov Modeling, is based on a simple idea: multidimensional distribution is *composed* from a system of low-dimensional (oligodimensional) distributions by repetitive application of a special operator of composition. This is also the reason why the models are called *compositional models*. In several papers, in which the properties of the operator and models were studied [3, 4, 5], it was shown (among others) that these models are, in a way, equivalent to Bayesian networks. Roughly speaking, *any multidimensional distribution representable by a Bayesian network can also be represented with approximately the same number of parameters (probabilities) in the form of a compositional models, and vice versa.*

Though Bayesian networks and compositional models represent the same class of distributions, they do not do it in the same way. Bayesian networks use *conditional distributions* whereas compositional models consist of *unconditional distributions*. Naturally, both types of models bear the same information but whilst some marginal distributions are explicitly expressed in compositional models, it may happen that their computation from a corresponding Bayesian network is rather computationally expensive. Therefore it appears that some of computational procedures designed for compositional models are (algorithmically) simpler than their Bayesian network counterparts.

The goal of this paper is to show that the operator of composition can also be introduced for belief functions. Moreover, we will show that it inherits the basic properties of its probabilistic pre-image and therefore it will enable us to introduce compositional models for multidimensional belief functions.

We will see that this approach enables us to represent, let us say, a 15-dimensional belief function as a sequence of 3 or 4-dimensional belief functions. Whilst representation of a 15-dimensional belief function is completely impossible (it would require in binary case $2^{2^{15}} = 2^{32k}$ numbers), representation of a 4-dimensional belief function requires only $2^{2^4} = 2^{16} = 64k$ numbers and therefore a model consisting of twelve 4-dimensional belief functions requires "only" $12 \times 2^{16} = 768k$ values.

Let us stress at the very beginning that this paper is the first one dealing with compositional models for belief functions. At this moment, we do not know what is the connection of the introduced operator of composition to different concepts of conditioning (and conditional independence) introduced for belief functions. The reader should realize that composition defined in this paper is different from that defined by Shenoy in [7]. His composition meets the requirements given by Shenoy's axioms (commutativity, associativity and distributivity) neither of which is met by the composition defined here. Therefore we do not know to what extent his principles of local computations are applicable to our model. This is one of many important open problems, some of which will be mentioned in Conclusions.

The reader familiar with the literature on belief functions is accustomed to the conjunctive rule of combination. Ben Yaghlane et al. [2] apply this rule to the set of marginal and conditional belief functions with the goal to compute a joint belief function in a way analogous to Bayesian networks (so-called Belief Chain Rule). This type of operation again substantially differs from the composition considered in this paper; the conjunctive rule of combination is commutative and associative. Moreover, in older papers, Xu and Smets consider only 2-dimensional belief functions, see e.g. [10].

Though the present paper is a contribution to belief function theory, we will not use the term of *belief function* any more in this paper. We are convinced that it will make the paper more legible for the reader when we will restrict our considerations to *basic belief assignments*, only. Therefore we will define a composition of basic assignments and show how to compose a sequence of simple basic assignments to get an assignment corresponding to a multidimensional belief function.

The contribution is organized as follows. In Section 2 we summarize basic notions, notation and introduce the operator of composition. Its basic properties can be found in Section 3, while Section 4 is devoted to more advanced properties. Finally, in Section 5 we introduce the notion of so-called *p*erfect sequences and demonstrate their importance.

2 Notation

Consider a finite index set $N = \{1, 2, \ldots, n\}$ and finite sets $\{\mathbf{X}_i\}_{i \in N}$. In this text we will consider *multidimensional frame of discernment*

$$\Omega = \mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

and its *subframes*. For $K \subset N$, $\mathbf{X}_K$ denotes a Cartesian product of those $\mathbf{X}_i$, for which $i \in K$:

$$\mathbf{X}_K = \bigtimes_{i \in K} \mathbf{X}_i.$$

A *projection* of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{X}_N$ into $\mathbf{X}_K$ will be denoted $x^{\downarrow K}$, i.e. for $K = \{i_1, i_2, \ldots, i_\ell\}$

$$x^{\downarrow K} = (x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}) \in \mathbf{X}_K.$$

Analogously, for $K \subset L \subseteq N$ and $A \subset \mathbf{X}_L$, $A^{\downarrow K}$ will denote a *projection* of A into $\mathbf{X}_K$:

$$A^{\downarrow K} = \{y \in \mathbf{X}_K | \exists x \in A : y = x^{\downarrow K}\}.$$

Let us remark that we do not exclude situations when $K = \emptyset$. In this case $A^{\downarrow \emptyset} = \emptyset$.

In addition to the projection, in this text we will need also the opposite operation which will be called extension. By an *extension* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ we will understand a set

$$A \otimes B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \ \ \& \ \ x^{\downarrow L} \in B\}.$$

Consider a *basic* (*probability or belief*) *assignment* (or just assignment) m on $\mathbf{X}_N$, i.e.

$$m : \mathcal{P}(\mathbf{X}_N) \longrightarrow [0, 1]$$

for which $\sum_{A \subseteq \mathbf{X}_N} m(A) = 1$. For each $K \subset N$ its *marginal basic assignment* is defined (for each $B \subseteq \mathbf{X}_K$):

$$m^{\downarrow K}(B) = \sum_{A \subseteq \mathbf{X}_N : A^{\downarrow K} = B} m(A).$$

Having two basic assignments m_1 and m_2 on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively (we assume that $K, L \subseteq N$), we say that these assignments are *projective* if

$$m_1^{\downarrow K \cap L} = m_2^{\downarrow K \cap L},$$

which occurs if and only if there exists a basic assignment m on $\mathbf{X}_{K \cup L}$ such that both m_1 and m_2 are marginal assignments of m.

Now, let us start considering how to define composition of two basic assignments. Consider two sets $K, L \subset N$. At this moment we do not pose any restrictions on K and L; they may be but need not be disjoint, one may be subset of the other. We even admit that one or both of them are empty[1]. Let m_1 and m_2 be basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively.

Our goal is to define new basic assignment, denoted $m_1 \triangleright m_2$, which will be defined on $\mathbf{X}_{K \cup L}$ and will contain all of the information contained in m_1 and as much as possible of information of m_2 (for the exact meaning see properties (iii) and (iv) of Lemma 1). The required property is met by the following definition.

Definition 1 For two arbitrary basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ a *composition* $m_1 \triangleright m_2$ is defined for all $C \subseteq \mathbf{X}_{K \cup L}$ by one of the following expressions:

[**a**] if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) > 0$ and $C = C^{\downarrow K} \otimes C^{\downarrow L}$ then

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L})};$$

[**b**] if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) = 0$ and $C = C^{\downarrow K} \times \mathbf{X}_{L \setminus K}$ then

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K});$$

[1] Notice that basic assignment m on $\mathbf{X}_\emptyset$ is defined $m(\emptyset) = 1$. Let us note that this is the only case where we accept $m(\emptyset) > 0$, otherwise $m(\emptyset) = 0$ according to the classical definitions of basic assignment and belief function, see [6].

[**c**] in all other cases

$$(m_1 \triangleright m_2)(C) = 0.$$

Remark Notice what this definition yields in the following degenerate situations:

- if $K \cap L = \emptyset$ then $m_1 \triangleright m_2 = m_1 \cdot m_2$ (recall that $m_2^{\downarrow\emptyset}(\emptyset) = 1$) — for details regarding this situation see Example 1;
- if $K \supseteq L$ then $m_1 \triangleright m_2 = m_1$.

3 Basic properties of composition

Lemma 1 *For arbitrary two basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ the following properties hold true:*

(i) $m_1 \triangleright m_2$ *is a basic assignment on* $\mathbf{X}_{K\cup L}$.

(ii) $(m_1 \triangleright m_2)^{\downarrow K} = m_1$.

(iii) $m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$.

(iv) *If $K \subseteq L$ then $m_2^{\downarrow K} \triangleright m_2 = m_2$.*

Proof. Let us first prove that for any $B \subseteq \mathbf{X}_K$

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} (m_1 \triangleright m_2)(A) = m_1(B). \qquad (1)$$

Since, due to Definition 1, $(m_1 \triangleright m_2)(C) = 0$ for any $C \subseteq \mathbf{X}_{K\cup L} \setminus (\mathbf{X}_K \otimes \mathbf{X}_L)$ (in other words for $C \neq C^{\downarrow K} \otimes C^{\downarrow L}$) we see that

$$\begin{aligned}\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} &(m_1 \triangleright m_2)(A) \\ &= \sum_{A\subseteq \mathbf{X}_K\otimes\mathbf{X}_L:A^{\downarrow K}=B} (m_1 \triangleright m_2)(A) \\ &= \sum_{C\subseteq \mathbf{X}_L:C^{\downarrow K\cap L}=B^{\downarrow K\cap L}} (m_1 \triangleright m_2)(B \otimes C).\end{aligned}$$

To prove formula (1), we have to distinguish two situations depending on the value of $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})$. If this value is positive then

$$\begin{aligned}\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} &(m_1 \triangleright m_2)(A) \\ &= \sum_{C\subseteq \mathbf{X}_L:C^{\downarrow K\cap L}=B^{\downarrow K\cap L}} \frac{m_1(B)\cdot m_2(C)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} \\ &= \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} \sum_{C\subseteq \mathbf{X}_L:C^{\downarrow K\cap L}=B^{\downarrow K\cap L}} m_2(C) \\ &= \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) \\ &= m_1(B).\end{aligned}$$

If $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = 0$ then, according to Definition 1, there exists only one $A \subseteq \mathbf{X}_{K\cup L}$ for which $A^{\downarrow K} = B$ such that $(m_1 \triangleright m_2)(A)$ may be positive; namely $A = B \times \mathbf{X}_{L\setminus K}$. Therefore

$$\begin{aligned}\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} &(m_1 \triangleright m_2)(A) \\ &= (m_1 \triangleright m_2)(B \times \mathbf{X}_{L\setminus K}) \\ &= m_1(B),\end{aligned}$$

Thus having proved that equality (1) holds true let us start proving assertions (i) – (iv).

ad (i) To prove that $m_1 \triangleright m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$ we have to show that for each $A \subseteq \mathbf{X}_{K\cup L}$ value $(m_1 \triangleright m_2)(A)$ is nonnegative (which is evident) and that the sum of all these values equals 1. The latter holds true, too, because (using equality (1))

$$\begin{aligned}\sum_{A\subseteq \mathbf{X}_{K\cup L}} &(m_1 \triangleright m_2)(A) \\ &= \sum_{B\subseteq \mathbf{X}_K} \sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} (m_1 \triangleright m_2)(A) \\ &= \sum_{B\subseteq \mathbf{X}_K} m_1(B) = 1.\end{aligned}$$

ad (ii) The formula is another form of equality (1).

ad (iii) Let us first prove

$$m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L} \implies m_1 \triangleright m_2 = m_2 \triangleright m_1.$$

Consider any $A \subseteq \mathbf{X}_{K\cup L}$. If $A \not\subseteq \mathbf{X}_K \otimes \mathbf{X}_L$ then both $(m_1 \triangleright m_2)(A)$ and $(m_2 \triangleright m_1)(A)$ equal 0. Therefore we have to prove the implication only for $A \subseteq \mathbf{X}_K \otimes \mathbf{X}_L$. If $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ then

$$\begin{aligned}(m_1 \triangleright m_2)(A) &= \frac{m_1(A^{\downarrow K})\cdot m_2(A^{\downarrow L})}{m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L})} \\ &= \frac{m_1(A^{\downarrow K})\cdot m_2(A^{\downarrow L})}{m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L})} \\ &= (m_2 \triangleright m_1)(A).\end{aligned}$$

In opposite when $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$, both $m_1(A^{\downarrow K})$ and $m_2(A^{\downarrow L})$ must equal 0 and therefore (according to Definition 1) $(m_1 \triangleright m_2)(A) = (m_2 \triangleright m_1)(A) = 0$.

To prove the other side of the equivalence (i.e. $m_1 \triangleright m_2 = m_2 \triangleright m_1$ implies $m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$) it is enough to realize that if $m_1^{\downarrow K\cap L} \neq m_2^{\downarrow K\cap L}$ then also $m_1 \triangleright m_2 \neq m_2 \triangleright m_1$ because, due to already proved (item ii) of this assertion, $m_1^{\downarrow K\cap L} = (m_1 \triangleright m_2)^{\downarrow K\cap L}$ and $m_2^{\downarrow K\cap L} = (m_2 \triangleright m_1)^{\downarrow K\cap L}$.

Table 1: Basic assignments m_1 and m_2.

$A \subseteq \mathbf{X}_1$	$m_1(A)$
$\{a_1\}$	0.2
$\{b_1\}$	0.3
$\{a_1b_1\}$	0.5

$A \subseteq \mathbf{X}_2$	$m_2(A)$
$\{a_2\}$	0.6
$\{b_2\}$	0
$\{a_1b_2\}$	0.4

Table 2: Basic assignment $m_1 \triangleright m_2$.

$C \subseteq \mathbf{X}_{\{1,2\}}$	$C = C^{\downarrow\{1\}} \otimes C^{\downarrow\{2\}}$	$(m_1 \triangleright m_2)(C)$
$\{a_1a_2\}$	$\{a_1\} \otimes \{a_2\}$	0.12
$\{a_1b_2\}$	$\{a_1\} \otimes \{b_2\}$	0
$\{b_1a_2\}$	$\{b_1\} \otimes \{a_2\}$	0.18
$\{b_1b_2\}$	$\{b_1\} \otimes \{b_2\}$	0
$\{a_1a_2, a_1b_2\}$	$\{a_1\} \otimes \mathbf{X}_2$	0.08
$\{a_1a_2, b_1a_2\}$	$\mathbf{X}_1 \otimes \{a_2\}$	0.3
$\{a_1a_2, b_1b_2\}$		0
$\{a_1b_2, b_1a_2\}$		0
$\{a_1b_2, b_1b_2\}$	$\mathbf{X}_1 \otimes \{b_2\}$	0
$\{b_1a_2, b_1b_2\}$	$\{b_1\} \otimes \mathbf{X}_2$	0.12
$\{a_1a_2, a_1b_2, b_1a_2\}$		0
$\{a_1a_2, a_1b_2, b_1b_2\}$		0
$\{a_1a_2, b_1a_2, b_1b_2\}$		0
$\{a_1b_2, b_1a_2, b_1b_2\}$		0
$\left\{\begin{matrix} a_1a_2, a_1b_2 \\ b_1a_2, b_1b_2 \end{matrix}\right\}$	$\mathbf{X}_1 \otimes \mathbf{X}_2$	0.2

ad (iv) This property follows directly from previously proved items (iii) and (ii). ∎

Let us now illustrate the operator of composition and its properties by two examples. The first shows what happens when $K \cap L = \emptyset$, the other demonstrates non-commutativity of the operator.

Example 1 Consider two basic assignments m_i (for $i = 1, 2$) on $\mathbf{X}_i = \{a_i, b_i\}$ specified in Table 1.[2] Since, in this case, $K \cap L$ is empty (recall that $m_2^{\downarrow\emptyset}(\emptyset) = 1$), composition simplifies to the expression

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow\{1\}}) \cdot m_2(C^{\downarrow\{2\}}).$$

Using Table 2, where the values of $m_1 \triangleright m_2$ are presented, the reader can easily check that $m_1 = (m_1 \triangleright m_2)^{\downarrow\{1\}}$, and since m_1 and m_2 are trivially projective also $m_2 = (m_1 \triangleright m_2)^{\downarrow\{2\}}$. ♦

[2]Let us note that, for the sake of simplicity, we use in examples $x_1 \dots x_n$ instead of $(x_1, \dots, x_n)$.

Example 2 Let for $i = 1, 2, 3$, $\mathbf{X}_i = \{a_i, b_i\}$ and let us consider the following basic assignments m_1 and m_2 on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively:

$$\begin{aligned} m_1(\mathbf{X}_1 \times \{a_2\}) &= 0.4, \\ m_1(\mathbf{X}_1 \times \mathbf{X}_2) &= 0.6, \\ m_2(\mathbf{X}_2 \times \{a_3\}) &= 0.5, \\ m_2(\mathbf{X}_2 \times \mathbf{X}_3) &= 0.5, \end{aligned}$$

the values of both basic assignments m_1 and m_2 on the remaining subsets being zero. From Definition 1 (case [a]) one can immediately see that both $(m_1 \triangleright m_2)(A)$ and $(m_2 \triangleright m_1)(A)$ can be positive only for those $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ for which

$$A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{a_2\} \quad \text{or} \quad A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \mathbf{X}_2,$$

and

$$A^{\downarrow\{2,3\}} = \mathbf{X}_2 \times \{a_3\} \quad \text{or} \quad A^{\downarrow\{2,3\}} = \mathbf{X}_2 \times \mathbf{X}_3.$$

There are only two such sets

$$A_1 = \mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\} \quad \text{and} \quad A_2 = \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3.$$

For these sets we get

$$\begin{aligned} &(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\}) \\ &\qquad = \tfrac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \{a_3\})}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} \\ &\qquad = \tfrac{0.6 \cdot 0.5}{1} = 0.3, \end{aligned}$$

$$\begin{aligned} &(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) \\ &\qquad = \tfrac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \mathbf{X}_3)}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} \\ &\qquad = \tfrac{0.6 \cdot 0.5}{1} = 0.3, \end{aligned}$$

and similarly

$$\begin{aligned} &(m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\}) \\ &\qquad = \tfrac{m_2(\mathbf{X}_2 \times \{a_3\}) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} \\ &\qquad = \tfrac{0.5 \cdot 0.6}{0.6} = 0.5, \end{aligned}$$

$$\begin{aligned} &(m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) \\ &\qquad = \tfrac{m_2(\mathbf{X}_2 \times \mathbf{X}_3) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} \\ &\qquad = \tfrac{0.5 \cdot 0.6}{0.6} = 0.5. \end{aligned}$$

From case [b] of Definition 1 we will get yet another focal element for $m_1 \triangleright m_2$, namely

$$A_3 = \mathbf{X}_1 \times \{a_2\} \times \mathbf{X}_3,$$

Table 3: Composed basic assignments.

	$(m_1 \triangleright m_2)(A)$	$(m_2 \triangleright m_1)(A)$
A_1	0.3	0.5
A_2	0.3	0.5
A_3	0.4	0

for which

$$A_3^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{a_2\} \quad \text{and} \quad A_3^{\downarrow\{3\}} = \mathbf{X}_3.$$

Since $m_2^{\downarrow\{2\}}(A_3^{\downarrow\{2\}}) = 0$ and $A_3^{\downarrow\{3\}} = \mathbf{X}_3$ we get

$$\begin{aligned}(m_1 \triangleright m_2)(\mathbf{X}_1 \times \{a_2\} \times \mathbf{X}_3) &= m_1(\mathbf{X}_1 \times \{a_2\}) \\ &= 0.4.\end{aligned}$$

Notice that there does not exist such a focal element for $m_2 \triangleright m_1$, as $m_1^{\downarrow\{2\}}(A_3^{\downarrow\{2\}}) > 0$.

Both the composed basic assignments $m_1 \triangleright m_2$ and $m_2 \triangleright m_1$ are outlined in Table 3 (recall once more that for all other $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ different from those included in Table 3 both assignments equal to 0). ♦

As said in the Introduction, operator of composition was originally introduced in probability theory. A basic assignment m degenerates into a probability distribution if all its focal elements are singletons (in other words: $m(A) > 0 \implies |A| = 1$).

In agreement with [6] we will call such assignments *Bayesian basic assignments*. It would be strange if the operator of composition we have introduced in this paper would not coincide with the probabilistic one if applied to Bayesian basic assignments. Fortunately, it is not the case.

Lemma 2 *Let m_1 and m_2 be Bayesian basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively, for which*

$$m_2^{\downarrow K\cap L}(A) = 0 \implies m_1^{\downarrow K\cap L}(A) = 0 \qquad (2)$$

for any $A \subseteq \mathbf{X}_{K\cup L}$. Then $m_1 \triangleright m_2$ is a Bayesian basic assignment.

Proof. To prove that a basic assignment $m_1 \triangleright m_2$ is Bayesian, it is enough to show that if $A \subseteq \mathbf{X}_{K\cup L}$ is not a singleton then $(m_1 \triangleright m_2)(A) = 0$.

Consider any $A \subseteq \mathbf{X}_{K\cup L}$, and two different elements $x, y \in A$. Since $x \neq y$ then either $x^{\downarrow K} \neq y^{\downarrow K}$ or $x^{\downarrow L} \neq y^{\downarrow L}$ (or both). Therefore either $A^{\downarrow K}$ or $A^{\downarrow L}$ is not a singleton and therefore $m_1(A^{\downarrow K}) \cdot m_2(A^{\downarrow L}) = 0$. This means that if $m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ then, due to Definition 1, $(m_1 \triangleright m_2)(A) = 0$.

If $m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$ then, because we assume the validity of implication (2), $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$ and therefore also $m_1(A^{\downarrow K}) = 0$. Therefore, according to Definition 1, $(m_1 \triangleright m_2)(A) = 0$, too. ∎

Remark The reader should however notice that the definition of the operator of composition for Bayesian basic assignments is not fully equivalent to the definition of composition for probabilistic distributions. They equal to each other only in case that the probabilistic version is defined. This is anchored in Lemma 2 by assuming the implication (2). In case it does not hold, the probabilistic operator is not defined whilst its belief version introduced in this paper is always defined. Nevertheless, in this case, the result is not a Bayesian assignment. We shall illustrate it by a simple example.

Example 3 Let $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ be as in the previous example and consider the following Bayesian basic assignments m_1 and m_2 on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively:

$$\begin{aligned}m_1(\{a_1a_2\}) &= m_1(\{a_1b_2\}) \\ &= m_1(\{b_1a_2\}) = m_1(\{b_1b_2\}) = 0.25, \\ m_2(\{a_2a_3\}) &= m_2(\{a_2b_3\}) = 0.5, \\ m_2(\{b_2a_3\}) &= m_2(\{b_2b_3\}) = 0.\end{aligned}$$

Let us compute $m_1 \triangleright m_2$ for singletons $\{x_1x_2x_3\} \in \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. If $x_2 = a_2$ then

$$\begin{aligned}&(m_1 \triangleright m_2)(\{x_1a_2x_3\}) \\ &\qquad = \tfrac{m_1(\{x_1a_2\}) \cdot m_2(\{a_2x_3\})}{m_2^{\downarrow 2}(\{a_2\})} \\ &\qquad = \tfrac{0.25 \cdot 0.5}{1} = 0.125.\end{aligned}$$

For a singleton $\{x_1b_2x_3\}$ we get

$$(m_1 \triangleright m_2)(\{x_1b_2x_3\}) = 0,$$

because $m_2^{\downarrow 2}(\{b_2\}) = 0$. In this case, however, we get

$$\begin{aligned}(m_1 \triangleright m_2)(\{x_1b_2\} \times \mathbf{X}_3) &= m_1(\{x_1b_2\}) \\ &= 0.25.\end{aligned}$$

This means that in this case there are 6 focal elements of $m_1 \triangleright m_2$, namely 4 singletons:

$$\{x_1a_2x_3\}, \text{ for } x_1 \in \mathbf{X}_1, x_3 \in \mathbf{X}_3,$$

and 2 two-element sets

$$\{x_1b_2\} \times \mathbf{X}_3, \text{ for } x_1 \in \mathbf{X}_1.$$

Let us remark that in contrast to $m_1 \triangleright m_2$, $m_2 \triangleright m_1$ is a Bayesian basic assignment, because whenever

Table 4: Basic assignments m_1 and m_2.

$A \subseteq \mathbf{X}_1$	$m_1(A)$
$\{a_1\}$	0.5
$\{a_1, b_1\}$	0.5

$A \subseteq \mathbf{X}_2$	$m_2(A)$
$\{a_2\}$	0.4
$\{a_2, b_2\}$	0.6

$m_1^{\downarrow\{2\}}(x_2) = 0$ then also $m_2^{\downarrow\{2\}}(x_2) = 0$. Basic assignment $m_1 \triangleright m_2$ has 4 focal elements:

$$\begin{aligned}&(m_2 \triangleright m_1)(\{a_1a_2a_3\})\\ &\qquad = (m_2 \triangleright m_1)(\{a_1a_2b_3\})\\ &\qquad = (m_2 \triangleright m_1)(\{b_1a_2a_3\})\\ &\qquad = (m_2 \triangleright m_1)(\{b_1a_2b_3\}) = 0.25. \qquad \blacklozenge\end{aligned}$$

Remark In Examples 2 and 3 we showed that the operator of composition is not commutative. From the following example we shall see that this operator is neither associative.

Example 4 Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be as in previous examples and let us consider the following three basic assignments m_1, m_2 defined on $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, as suggested in Table 4 and m_3 have only one focal element, namely

$$m_3(\mathbf{X}_1 \times \mathbf{X}_2) \quad = \quad 1.$$

Then

$$\begin{aligned}(m_1 \triangleright m_2)(\{a_1a_2\}) &= 0.2,\\ (m_1 \triangleright m_2)(\{a_1\} \times \mathbf{X}_2) &= 0.3,\\ (m_1 \triangleright m_2)(\mathbf{X}_1 \times \{a_2\}) &= 0.2,\\ (m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2) &= 0.3,\end{aligned}$$

due to Definition 1 (the values on remaining sets being again zero) and $(m_1 \triangleright m_2) \triangleright m_3 = m_1 \triangleright m_2$ according to Lemma 1 property (iv). On the other hand

$$\begin{aligned}(m_2 \triangleright m_3)(\mathbf{X}_1 \times \{a_2\}) &= 0.4,\\ (m_2 \triangleright m_3)(\mathbf{X}_1 \times \mathbf{X}_2) &= 0.6.\end{aligned}$$

Now, computing $m_1 \triangleright (m_2 \triangleright m_3)$ we obtain

$$\begin{aligned}(m_1 \triangleright (m_2 \triangleright m_3))(\{a_1\} \times \mathbf{X}_2) &= 0.5,\\ (m_1 \triangleright (m_2 \triangleright m_3))(\mathbf{X}_1 \times \{a_2\}) &= 0.2,\\ (m_1 \triangleright (m_2 \triangleright m_3))(\mathbf{X}_1 \times \mathbf{X}_2) &= 0.3,\end{aligned}$$

which evidently differs from $(m_1 \triangleright m_2) \triangleright m_3$ (see Table 5). $\blacklozenge$

Table 5: Composed basic assignments.

	$(m_1 \triangleright m_2) \triangleright m_3$	$m_1 \triangleright (m_2 \triangleright m_3)$
$\{a_1a_2\}$	0.2	0
$\{a_1\} \times \mathbf{X}_2$	0.3	0.5
$\mathbf{X}_1 \times \{a_2\}$	0.2	0.2
$\mathbf{X}_1 \times \mathbf{X}_2$	0.3	0.3

4 Advanced properties of composition

In this section we are going to study properties which were proved for probabilistic version of the operator of composition and which are applied when proving important theorems regarding compositional models. Unless expressed explicitly otherwise in this section we will assume m_1, m_2, m_3 be basic assignments on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}, \mathbf{X}_{K_3}$, respectively.

Lemma 3 *Let m_1, m_2, m_3 be basic assignments on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}, \mathbf{X}_{K_3}$, respectively. If $K_1 \supseteq (K_2 \cap K_3)$ then*

$$(m_1 \triangleright m_2) \triangleright m_3 = (m_1 \triangleright m_3) \triangleright m_2. \tag{3}$$

Proof. The goal is to prove that for any $C \subseteq \mathbf{X}_{K_1 \cup K_2 \cup K_3}$

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = ((m_1 \triangleright m_3) \triangleright m_2)(C). \tag{4}$$

We will have to distinguish five special cases.

A. $C \neq C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes C^{\downarrow K_3}$.
This is the simplest situation because in this case both sides of formula (4) equal 0 due to Definition 1 (case [c]).

B. $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes C^{\downarrow K_3}$ & $m_2^{\downarrow K_1 \cap K_2}(C^{\downarrow K_1 \cap K_2})$, $m_3^{\downarrow K_1 \cap K_3}(C^{\downarrow K_1 \cap K_3}) > 0$.
In this case it is enough to realize that (under the given assumptions) $K_3 \cap (K_1 \cup K_2) = K_3 \cap K_1$ and, analogously, $K_2 \cap (K_1 \cup K_3) = K_2 \cap K_1$. Then we see that both sides of formula (4) again coincide:

$$\begin{aligned}&((m_1 \triangleright m_2) \triangleright m_3)(C)\\ &\qquad = \frac{m_1(C^{\downarrow K_1}) \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_2 \cap K_1}(C^{\downarrow K_2 \cap K_1})}\\ &\qquad \quad \cdot \frac{m_3(C^{\downarrow K_3})}{m_3^{\downarrow K_3 \cap (K_1 \cup K_2)}(C^{\downarrow K_3 \cap (K_1 \cup K_2)})},\end{aligned}$$

$$\begin{aligned}&((m_1 \triangleright m_3) \triangleright m_2)(C)\\ &\qquad = \frac{m_1(C^{\downarrow K_1}) \cdot m_3(C^{\downarrow K_3})}{m_3^{\downarrow K_3 \cap K_1}(C^{\downarrow K_3 \cap K_1})}\\ &\qquad \quad \cdot \frac{m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_2 \cap (K_1 \cup K_3)}(C^{\downarrow K_2 \cap (K_1 \cup K_3)})}.\end{aligned}$$

C. $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes C^{\downarrow K_3}$,
$m_2^{\downarrow K_1 \cap K_2}(C^{\downarrow K_1 \cap K_2}) > 0 = m_3^{\downarrow K_1 \cap K_3}(C^{\downarrow K_1 \cap K_3})$.
In this case, if $C^{\downarrow K_3 \setminus K_1} \neq \mathbf{X}_{K_3 \setminus K_1}$ then both sides of formula (4) equal 0, because, due to Definition 1, both assignments $m_1 \triangleright m_2$ and $(m_1 \triangleright m_3) \triangleright m_2$ equal 0. Therefore consider $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes \mathbf{X}_{K_3 \setminus K_1}$. For this we get from Definition 1

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = (m_1 \triangleright m_2)(C^{\downarrow K_1 \cup K_2}).$$

For the right-hand side of formula (4) we get

$$(m_1 \triangleright m_3)(C^{\downarrow K_1 \cup K_3}) = m_1(C^{\downarrow K_1})$$

and therefore

$$((m_1 \triangleright m_3) \triangleright m_2)(C) = (m_1 \triangleright m_2)(C^{\downarrow K_1 \cup K_2}).$$

D. $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes C^{\downarrow K_3}$,
$m_2^{\downarrow K_1 \cap K_2}(C^{\downarrow K_1 \cap K_2}) = 0 < m_3^{\downarrow K_1 \cap K_3}(C^{\downarrow K_1 \cap K_3})$.
The proof is analogous to that under item C.

E. $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2} \otimes C^{\downarrow K_3}$,
$m_2^{\downarrow K_1 \cap K_2}(C^{\downarrow K_1 \cap K_2}) = 0 = m_3^{\downarrow K_1 \cap K_3}(C^{\downarrow K_1 \cap K_3})$.
It is obvious from Definition 1 that both sides of formula (4) equal 0 for all C but for $C = C^{\downarrow K_1} \otimes \mathbf{X}_{K_2 \setminus K_1} \otimes \mathbf{X}_{K_3 \setminus K_1}$. For this special case, however,

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = m_1(C^{\downarrow K_1}),$$
$$((m_1 \triangleright m_3) \triangleright m_2)(C) = m_1(C^{\downarrow K_1}). \qquad \blacksquare$$

Lemma 4 *Let m_1, m_2 be basic assignments on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}$, respectively. If $K_1 \cup K_2 \supseteq L \supseteq K_1$ then*

$$(m_1 \triangleright m_2)^{\downarrow L} = m_1 \triangleright m_2^{\downarrow K_2 \cap L}.$$

Proof. Consider first $B \subseteq \mathbf{X}_L$ such that $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) > 0$. For this B we get

$$\begin{aligned}
&(m_1 \triangleright m_2)^{\downarrow L}(B) \\
&= \sum_{A \subseteq \mathbf{X}_{K_1 \cup K_2} : A^{\downarrow L} = B} (m_1 \triangleright m_2)(A) \\
&= \sum_{A \subseteq \mathbf{X}_{K_1} \otimes \mathbf{X}_{K_2} : A^{\downarrow L} = B} (m_1 \triangleright m_2)(A) \\
&= \sum_{A \subseteq \mathbf{X}_{K_1} \otimes \mathbf{X}_{K_2} : A^{\downarrow L} = B} \frac{m_1(A^{\downarrow K_1}) \cdot m_2(A^{\downarrow K_2})}{m_2^{\downarrow K_1 \cap K_2}(A^{\downarrow K_1 \cap K_2})} \\
&= \sum_{C \subseteq \mathbf{X}_{K_2} : C^{\downarrow L \cap K_2} = B^{\downarrow L \cap K_2}} \frac{m_1(B^{\downarrow K_1}) \cdot m_2(C)}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} \\
&= \frac{m_1(B^{\downarrow K_1})}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} \sum_{C \subseteq \mathbf{X}_{K_2} : C^{\downarrow L \cap K_2} = B^{\downarrow L \cap K_2}} m_2(C) \\
&= \frac{m_1(B^{\downarrow K_1}) m_2^{\downarrow L \cap K_2}(B^{\downarrow L \cap K_2})}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} \\
&= (m_1 \triangleright m_2^{\downarrow L \cap K_2})(B).
\end{aligned}$$

If $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) = 0$ for some $B \subseteq \mathbf{X}_L$, then there is only one $A \subseteq \mathbf{X}_{K_1 \cup K_2}$ such that $A^{\downarrow K_1} = B^{\downarrow K_1}$ for which $(m_1 \triangleright m_2)(A)$ may be positive, namely $A^* = B^{\downarrow K_1} \otimes \mathbf{X}_{K_2 \setminus K_1}$ with $(m_1 \triangleright m_2)(A^*) = m_1(B^{\downarrow K_1})$. Thus if $B = B^{\downarrow K_1} \otimes \mathbf{X}_{L \setminus K_1}$,

$$\begin{aligned}
(m_1 \triangleright m_2)^{\downarrow L}(B) &= \sum_{A \subseteq \mathbf{X}_{K_1 \cup K_2} : A^{\downarrow L} = B} (m_1 \triangleright m_2)(A) \\
&= (m_1 \triangleright m_2)(A^*) = m_1(B^{\downarrow K_1}) \\
&= (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(A^{*\downarrow L}) \\
&= (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(B).
\end{aligned}$$

If $B \neq B^{\downarrow K_1} \otimes \mathbf{X}_{L \setminus K_1}$ and $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) = 0$ then

$$(m_1 \triangleright m_2)^{\downarrow L}(B) = 0 = (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(B). \qquad \blacksquare$$

Lemma 5 *Let m_1, m_2 be basic assignments on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}$, respectively. If $K_1 \cup K_2 \supseteq L \supseteq K_1 \cap K_2$ then*

$$(m_1 \triangleright m_2)^{\downarrow L} = m_1^{\downarrow K_1 \cap L} \triangleright m_2^{\downarrow K_2 \cap L}.$$

Proof. We will compute the required marginal assignment in two steps. In the first step we will employ Lemma 4, then (iv) of Lemma 1 and finally Lemma 3:

$$\begin{aligned}
(m_1 \triangleright m_2)^{\downarrow K_1 \cup L} &= m_1 \triangleright m_2^{\downarrow K_2 \cap L} \\
&= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_1) \triangleright m_2^{\downarrow K_2 \cap L} \\
&= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_2^{\downarrow K_2 \cap L}) \triangleright m_1.
\end{aligned}$$

The last expression will be further marginalized with the help of Lemma 4 and afterwards the final form will be received with application of Lemma 3 and (iv) of Lemma 1.

$$\begin{aligned}
(m_1 \triangleright m_2)^{\downarrow L} &= \left((m_1^{\downarrow K_1 \cap K_2} \triangleright m_2^{\downarrow K_2 \cap L}) \triangleright m_1\right)^{\downarrow L} \\
&= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_2^{\downarrow K_2 \cap L}) \triangleright m_1^{\downarrow K_1 \cap L} \\
&= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_1^{\downarrow K_1 \cap L}) \triangleright m_2^{\downarrow K_2 \cap L} \\
&= m_1^{\downarrow K_1 \cap L} \triangleright m_2^{\downarrow K_2 \cap L}. \qquad \blacksquare
\end{aligned}$$

Lemma 6 *Let m_1, m_2 be basic assignments on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}$, respectively. Then*

$$m_1 \triangleright m_2 = m_1 \triangleright (m_1 \triangleright m_2)^{\downarrow K_2}.$$

Proof. Due to (ii) of Lemma 1 assignments m_1 and $(m_1 \triangleright m_2)^{\downarrow K_2}$ are projective and therefore (due to property (iii) of the same lemma) these arguments may be commuted

$$\begin{aligned}
m_1 \triangleright (m_1 \triangleright m_2)^{\downarrow K_2} &= (m_1 \triangleright m_2)^{\downarrow K_2} \triangleright m_1 \\
&= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_2) \triangleright m_1,
\end{aligned}$$

where the last modification is made on the basis of Lemma 5. The last expression meets the assumptions of Lemma 3 and therefore we can exchange second and third arguments, from which the required expression is got by application of (iv) of Lemma 1:

$$\begin{aligned}(m_1^{\downarrow K_1\cap K_2} \triangleright m_2) \triangleright m_1 &= (m_1^{\downarrow K_1\cap K_2} \triangleright m_1) \triangleright m_2 \\ &= m_1 \triangleright m_2. \qquad \blacksquare\end{aligned}$$

5 Compositional models

Now we are starting to consider repetitive application of the operator of composition with the goal to create a multidimensional model. Since the operator is neither commutative nor associative we have always to specify in which order the oligodimensional assignments are composed together. To make the formulas more lucid we will omit brackets in case that the operator is to be applied from left to right, i.e., in what follows

$$\begin{aligned}&m_1 \triangleright m_2 \triangleright m_3 \triangleright \ldots \triangleright m_{n-1} \triangleright m_n \\ &\qquad = (\ldots((m_1 \triangleright m_2) \triangleright m_3) \triangleright \ldots \triangleright m_{n-1}) \triangleright m_n.\end{aligned}$$

Moreover, we will always assume m_i be basic assignment on $\mathbf{X}_{K_i}$.

The reader familiar with some papers on probabilistic or possibilistic compositional models knows that one of the most important notions of this theory is that of a so-called *perfect sequence*, which will be now introduced also for a sequence of basic assignments.

Definition 2 A generating sequence of basic assignments m_1, $m_2, \ldots, m_n$ is called *perfect* if

$$\begin{aligned} m_1 \triangleright m_2 &= m_2 \triangleright m_1, \\ m_1 \triangleright m_2 \triangleright m_3 &= m_3 \triangleright (m_1 \triangleright m_2), \\ &\vdots \\ m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n &= m_n \triangleright (m_1 \triangleright \ldots \triangleright m_{n-1}).\end{aligned}$$

From the practical point of view it is also important to have a tool enabling us to recognize whether a generating sequence is perfect or not. For this one can take advantage of the following assertion.

Lemma 7 *A generating sequence $m_1, m_2, \ldots, m_n$ is perfect* iff *the pairs of basic assignments m_j and $(m_1 \triangleright \ldots \triangleright m_{j-1})$ are projective, i.e. if*

$$\begin{aligned}&m_j^{\downarrow K_j\cap(K_1\cup\ldots\cup K_{j-1})} \\ &\qquad = (m_1 \triangleright \ldots \triangleright m_{j-1})^{\downarrow K_j\cap(K_1\cup\ldots\cup K_{j-1})},\end{aligned}$$

for all $j = 2, 3, \ldots, n$.

Proof. This assertion is proved just by a multiple application of assertion (iii) of Lemma 1:

$$\begin{aligned}&m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow K_2\cap K_1} = m_2^{\downarrow K_2\cap K_1}, \\ &m_1 \triangleright m_2 \triangleright m_3 = m_3 \triangleright (m_1 \triangleright m_2) \\ &\qquad \iff (m_1 \triangleright m_2)^{\downarrow K_3\cap(K_1\cup K_2)} = m_3^{\downarrow K_3\cap(K_1\cup K_2)}, \\ &\qquad \vdots \\ &m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n = m_n \triangleright (m_1 \triangleright \ldots \triangleright m_{n-1}) \\ &\qquad \iff (m_1 \triangleright \ldots \triangleright m_{n-1})^{\downarrow K_n\cap(K_1\cup\ldots\cup K_{n-1})} \\ &\qquad\qquad = m_n^{\downarrow K_n\cap(K_1\cup\ldots\cup K_{n-1})}. \qquad \blacksquare\end{aligned}$$

From Definition 2 one can hardly see what are the properties of the perfect sequences; the main one is expressed by the following characterization theorem.

Theorem 1 *A generating sequence of basic assignments $m_1, m_2, \ldots, m_n$ is perfect* iff *all the assignments from this sequence are marginal to the composed basic assignment $m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n$:*

$$(m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n)^{\downarrow K_j} = m_j,$$

for all $j = 1, \ldots, m$.

Proof. The fact that all assignments m_j from a perfect sequence are marginals of $(m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n)$ follows from the fact that $(m_1 \triangleright \ldots \triangleright m_j)$ is marginal to $(m_1 \triangleright \ldots \triangleright m_n)$ (due to (ii) of Lemma 1) and m_j is marginal to $m_j \triangleright (m_1 \triangleright \ldots \triangleright m_{j-1}) = m_1 \triangleright \ldots \triangleright m_j$.

Suppose now that for all $j = 1, \ldots, n$, m_j are marginal assignments to $m_1 \triangleright \ldots \triangleright m_n$. It means that all the assignments from the sequence are pairwise projective, and that each m_j is projective with any marginal assignment of $m_1 \triangleright \ldots \triangleright m_n$, and consequently also with $m_1 \triangleright \ldots \triangleright m_{j-1}$. So we get that

$$\begin{aligned}&m_j^{\downarrow K_j\cap(K_1\cup\ldots\cup K_{j-1})} \\ &\qquad = (m_1 \triangleright \ldots \triangleright m_{j-1})^{\downarrow K_j\cap(K_1\cup\ldots\cup K_{j-1})}\end{aligned}$$

for all $j = 2, \ldots, n$, which is equivalent, due to Lemma 7, to the fact that $m_1, \ldots, m_n$ is perfect. $\blacksquare$

Graphical Markov models (or rather decomposable models) are recalled by the following (almost trivial) assertion, which resembles assertions concerning decomposable models.

Theorem 2 *Let a generating sequence of pairwise projective assignments $m_1, m_2, \ldots, m_n$ be such that $K_1, K_2, \ldots, K_n$ meets the well-known running intersection property:*

$$\forall j = 2, 3, \ldots, n \quad \exists \ell (1 \leq \ell < j) \quad \textit{such that} \quad K_j \cap (K_1 \cup \ldots \cup K_{j-1}) \subseteq K_\ell.$$

Then $m_1, m_2, \ldots, m_n$ is perfect.

Proof. Due to Lemma 7 it is enough to show that for each $j = 2, \ldots, n$ basic assignment m_j and the composed assignment $m_1 \triangleright \ldots \triangleright m_{j-1}$ are projective. Let us prove it by induction.

For $j = 2$ the required projectivity is guaranteed by the fact that we assume pairwise projectivity of all $m_1, \ldots, m_n$. So we have to prove it for general $j > 2$ under the assumption that the assertion holds for $j-1$, which means (due to Theorem 1) that all $m_1, m_2, \ldots, m_{j-1}$ are marginal to $m_1 \triangleright \ldots \triangleright m_{j-1}$. Since we assume that $K_1, \ldots, K_n$ meets the running intersection property, there exists $\ell \in \{1, 2, \ldots j-1\}$ such that $K_j \cap (K_1 \cup \ldots \cup K_{j-1}) \subseteq K_\ell$. Therefore $(m_1 \triangleright \ldots \triangleright m_{j-1})^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})}$ and $m_\ell^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})}$ are the same marginals of $m_1 \triangleright \ldots \triangleright m_{j-1}$ and therefore they have to equal to each other:

$$(m_1 \triangleright \ldots \triangleright m_{j-1})^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})} = m_\ell^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})}.$$

However we assume that m_j and m_ℓ are projective and therefore also

$$(m_1 \triangleright \ldots \triangleright m_{j-1})^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})} = m_j^{\downarrow K_j \cap (K_1 \cup \ldots \cup K_{j-1})}. \qquad \blacksquare$$

It should be stressed at this moment that running intersection property of $K_1, K_2, \ldots, K_n$ is a sufficient condition guaranteeing a perfectness of a generating sequence of pairwise projective assignments. By no means this condition is necessary as it will be shown in the following example.

Example 5 Simple example is given by two basic assignments m_1 and m_2 from Example 1 (recall that they are defined on $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and their values can be found in Table 1) and the third assignment $m_3 = m_1 \triangleright m_2$ (see Table 2). Considering sequence m_1, m_2, m_3, it is evident that $K_1 = \{1\}, K_2 = \{2\}, K_3 = \{1, 2\}$ do not meet the running intersection property. And yet the sequence m_1, m_2, m_3 is perfect because all the assignments are marginal (or equal) to $m_1 \triangleright m_2 \triangleright m_3$. Notice that if we chose any other basic assignment $\hat{m}_3$ on $\mathbf{X}_{\{1,2\}}$ different from $m_3 = m_1 \triangleright m_2$, the generating sequence $m_1, m_2, \hat{m}_3$ would not be perfect any more. So we see that perfectness of a sequence is not only a structural property connected with the properties of $K_1, K_2, \ldots, K_n$ but depends also on specific values of the respective basic assignments. ♦

The last assertion shows that each generating sequence defining a compositional model $m_1 \triangleright \ldots \triangleright m_n$ can be transformed into a perfect sequence. It means, any basic assignment representable by a generating sequence $m_1, m_2, \ldots, m_n$ can be represented also by a perfect sequence $\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_n$

Theorem 3 *For any generating sequence $m_1, m_2, \ldots, m_n$ the sequence $\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_n$ computed by the following process*

$$\begin{aligned}
\hat{m}_1 &= m_1, \\
\hat{m}_2 &= \hat{m}_1^{\downarrow K_2 \cap K_1} \triangleright m_2, \\
\hat{m}_3 &= (\hat{m}_1 \triangleright \hat{m}_2)^{\downarrow K_3 \cap (K_1 \cup K_2)} \triangleright m_3, \\
&\vdots \\
\hat{m}_n &= (\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_{n-1})^{\downarrow K_n \cap (K_1 \cup \ldots K_{n-1})} \triangleright m_n
\end{aligned}$$

is perfect and

$$m_1 \triangleright \ldots \triangleright m_n = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_n.$$

Proof. The perfectness of the sequence $\hat{m}_1, \ldots, \hat{m}_n$ follows immediately from Lemma 7 and from the definition of this sequence as

$$\hat{m}_i^{\downarrow K_i \cap (K_1 \cup \ldots \cup K_{i-1})} = (\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_{i-1})^{\downarrow K_i \cap (K_1 \cup \ldots \cup K_{i-1})}$$

yields projectivity of $(\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_{i-1})$ and $\hat{m}_i$.

Let us prove

$$m_1 \triangleright \ldots \triangleright m_n = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_n$$

by mathematical induction. Since $m_1 = \hat{m}_1$ by definition, it is enough to show that

$$m_1 \triangleright \ldots \triangleright m_i = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i$$

implies also

$$m_1 \triangleright \ldots \triangleright m_{i+1} = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_{i+1}.$$

In the following computations we will use the fact that due to Lemma 5

$$(\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i)^{\downarrow K_{i+1} \cap (K_1 \cup \ldots K_i)} \triangleright m_{i+1} = ((\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i) \triangleright m_{i+1})^{\downarrow K_{i+1}}$$

and afterwards we will employ Lemma 6.

$$\begin{aligned}
&\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_{i+1} \\
&\quad = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i \triangleright \\
&\qquad\qquad \Big((\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i)^{\downarrow K_{i+1} \cap (K_1 \cup \ldots K_i)} \triangleright m_{i+1} \Big) \\
&\quad = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i \triangleright \left((\hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i) \triangleright m_{i+1} \right)^{\downarrow K_{i+1}} \\
&\quad = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i \triangleright m_{i+1} = m_1 \triangleright \ldots \triangleright m_i \triangleright m_{i+1},
\end{aligned}$$

where the last modification is an application of the inductive assumption. ∎

6 Conclusions

Graphical Markov Models were designed to enable description of real-life problems by probabilistic models. Since we are getting into problems when coping with computational complexity of probabilistic models, all the more so problems naturally appear when applying belief function models, for which there do not exist distribution functions; we have to represent them by set functions defined on the whole power set of the frame of discernment $\Omega = \mathbf{X}_N$. So, inspired by the original probabilistic approach the paper is the first attempt to build up compositional models of multidimensional belief functions. We have defined belief function operator of composition manifesting all the main characteristics of its probabilistic pre-image. Even more, there is one point in which the belief function operator of composition is superior to the probabilistic one: thanks to the ability of belief functions to model total ignorance, the operator of composition is for basic assignments always defined, which is not the case in the probabilistic framework.

In the paper we have proved the basic properties of the operator necessary to introduce compositional models and their most important special case, perfect sequence models. Naturally, there are still many open problems to be solved. The most important one is a design of efficient computational procedures for this type of models. It is also necessary to clarify interrelations between the operator of composition and conditional independence. This problem is not easy because in the framework of belief functions there exist several notions corresponding to stochastic conditional independence.

At this moment we know very little about similarities and differences between the described compositional models and other multidimensional models such as [1, 2, 7], as well as about the relation between the compositional models developed for belief functions and those introduced in possibility theory [8, 9].

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