Multiparameter models: Probability distributions parameterized by random sets.

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Abstract

This paper is devoted to the construction of sets of joint probability measures for the case that the marginal sets of probability measures are generated by probability measures with uncertain parameters where the uncertainty of these parameters is modelled by random sets. Further we show how different conditions on the choice of the weights of the joint focal sets and on the probability measures associated to these sets lead to different sets of joint probability measures including the cases of strong independence, random set independence and unknown interaction.

Keywords. Random sets, lower and upper probabilities, sets of probability measures, parameterized probability measures, sets of joint probability measures, strong independence, random set independence, unknown interaction.

1 Introduction

Let a mapping

$$g : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R} : \ (x_1, \ldots, x_m) \mapsto g(x_1, \ldots, x_m)$$

be given. The variables x_k are assumed to be uncertain where the uncertainty is modelled by sets of probability measures for each variable separately. What we want to know is the lower and upper probabilities if the value $g(x)$, $x = (x_1, \ldots, x_m)$, is lower (or greater) than a certain value. Therefore we had to propagate the uncertainty of the variables x_k through this multivariate model g, c.f. [1].

As a short motivation we want to mention a few applications where this problem of propagating uncertain variables is arising:

Reliability analyis: In this case the above mapping g is the so called *failure function* where $g(x) \leq 0$ means failure and $g(x) > 0$ means no failure of buildings like bridges and tunnels in civil engineering; or of slopes and dams in geotechnical engineering. The aim is to describe the risk of failure, that means we want to have the upper probability $\overline{P}(\{g(x) \leq 0\})$ of failure. The variables x_k are parameters as elastic modulus E, angle of friction ϕ or flood heights.

Construction management: Here the values of $g(x)$ are costs or durations which should not exceed a certain bound a where the variables x_k are costs, durations or similar parameters as above. Then we want to have the upper probability $\overline{P}(\{g(x) \geq a\})$.

In most cases all these variables are not precisely known, especially parameters arsing in geotechnical engineering are only very vaguely known. In engineering there are several approaches used to describe the uncertainty of these variables: wellknown ones as *probability distributions* or *intervals* and more modern ones as *fuzzy sets* and *random sets*. The uncertainty of the variables is given separately and often modelled by different ways. So a unifying approach is needed to combine and propagate the different models of uncertainty trough the function g. This is provided by the concept of *sets of probability measures* where these sets are generated by random sets which are including the other three approaches (probability distributions, intervals and fuzzy sets).

In some applications the type of probability distributions to discribe the uncertainty of a variable is known, e.g. gaussian distributions, exponential distributions in queueing theory, etreme value distributions in flood risk analysis, but the parameters of these probability distributions are often only vaguely known. In these cases we have to model the uncertainty of the parameters of these distributions. So we introduce here the concept of sets of probability measures which are generated by *parameterized probability measures* where these parameters are uncertain and the uncertainty is described by random sets. All models of uncertainty mentioned before are special cases of this concept.

Since the uncertainty of the variables is given separately, we have to model the joint uncertainty, that means to construct the set of joint probability measures. There are certain ways to generate such sets, e.g. according to strong independence [2, 11] if we assume stochastically independence of the variables, or according to unknown interaction [2] if we do not know how the variables interact, or according to random set independence [3] since random sets are involved. These cases are already studied for sets of probability measures generated by random sets in [5, 6, 7, 8]. Here in this paper we extend this to sets of probability measures generated by parameterized probability measures with uncertain parameters.

To propagate the uncertainty through a multivariate model in a computational efficient way it is essential to make use of the structure of the random sets. We show how different conditions on the parts of this structure (on the choice of the weights of the joint focal sets and on the probability measures associated to these sets) lead to different sets of joint probability measures. But our goal is not to create new artificial types of sets of joint probability measures, but to get sets according to strong independence or unknown interaction by using the random set structure.

The plan of this paper is as follows:

Section 2 is devoted to random sets and the parameterization of probability measures by random sets in the univariate case. In Section 3 we construct sets of joint probability measures which are generated by probability measures which are parameterized by ordinary sets as preliminary work for Sec. 5. In Section 4 we recall from [5, 7] the general formulation for constructing sets of joint probability measures for the case where random sets are involved and list different conditions on choosing the weights of the joint focal sets and the probability measures associated to these sets. In Section 5 we show that some of these cases lead to strong independence, random set independence and unknown interaction.

2 Sets of probability measures generated by probability measures parameterized by random sets

We want to model the uncertainty about the value of a variable x by a convex set $\mathcal{K}$ of probability measures in the univariate case. Here in this paper we generate such sets $\mathcal{K}$ by a parameterized probability measure $p^{\,\theta}$ where $\theta = (\theta^1, \theta^2, \dots)$ are the parameters of the probability measure. These parameters are assumed to be uncertain. The uncertainty of θ is modelled by random sets. So we have to recall the concept of random sets and sets of probability measures generated by random sets. Further we need two different measurable spaces: $(\Omega, \mathcal{A})$ for the uncertain variable itself and $(\Theta, \mathfrak{A})$ for the uncertain parameters of the probability measures.

2.1 Random sets

First we want to model the uncertainty of a variable x by random sets. Let a measurable space $(\Omega, \mathcal{A})$ be given. A random set $(\mathcal{F}, m)$ [3, 4] consists of a finite class

$$\mathcal{F} = \{F^1, F^2, \dots, F^n\} \subseteq \mathcal{A}$$

of focal sets and of a weight function

$$m : \mathcal{F} \longrightarrow [0,1] : F \mapsto m(F)$$

with $\sum_{i=1}^{|\mathcal{F}|} m(F^i) = 1$ where $|\mathcal{F}|$ is the number of focal sets. Then the plausibility measure Pl or upper probability $\overline{P}$ of a set $A \in \mathcal{A}$ is defined by

$$\overline{P}(A) = \mathrm{Pl}(A) = \sum_{F^i \cap A \neq \varnothing} m(F^i)$$

and the belief measure Bel or lower probability $\underline{P}$ by

$$\underline{P}(A) = \mathrm{Bel}(A) = \sum_{F^i \subseteq A} m(F^i).$$

2.2 Sets of probability measures generated by random sets

The focal set F^i has the weight $m(F^i)$, but we do not know how this weight is distributed on the elements of the focal set which reflects the uncertainty modelled by a random set. Let

$$\mathcal{K}(F^i) := \{P : \ P(F^i) = 1\} \tag{1}$$

be the set of all probability measures "on" the focal set F^i. Then $m(F^i)\mathcal{K}(F^i)$ is the set of all possible distributions of the weight on the focal set. A convex set of probability measures $\mathcal{K}$ is generated by the random set $(\mathcal{F}, m)$ as follows [5]:

$$\mathcal{K} := \mathcal{K}(\mathcal{F}, m) := \sum_{i=1}^{|\mathcal{F}|} m(F^i)\,\mathcal{K}(F^i) := \tag{2}$$
$$= \left\{ P : \ P = \sum_{i=1}^{|\mathcal{F}|} m(F^i) P^i, \ P^i \in \mathcal{K}(F^i) \right\}.$$

This set $\mathcal{K}(\mathcal{F}, m)$ coincides with the set of probability measures defined by

$$\{P : \ \forall A \in \mathcal{A} : \mathrm{Bel}(A) \leq P(A) \leq \mathrm{Pl}(A)\},$$

c.f. [3, 4, 10].

Remark: There is a second approach of defining random sets: using multivalued mappings and measurable selections [9, 10]. This approach leads to a set $\mathcal{M}$ of probability measures which is a subset of $\mathcal{K}$ and which is not convex in general. The set of probability measures associated to the measurable selctions is $P_{\overline{\Omega}}(\Gamma) = \{P_X : \; X \in S(\Gamma)\}$, where $\Gamma : \overline{\Omega} \to \mathcal{A}$ is a multivalued mapping defined on a probability space $(\overline{\Omega}, \overline{\mathcal{A}}, P_{\overline{\Omega}})$. $S(\Gamma)$ is the set of measurable selections of Γ, that means the class of random variables $X : \overline{\Omega} \to \Omega$ with $X(\overline{\omega}) \in \Gamma(\overline{\omega})$.

Now let $P_X \in P(\Gamma)$, $X \in S(\Gamma)$, be given. Then

$$\begin{aligned} P_X(A) = P_{\overline{\Omega}}(X^{-1}(A)) &= \sum_{i=1}^{|\overline{\Omega}|} P_{\overline{\Omega}}(\{\overline{\omega}^i\}) \chi_A(X(\overline{\omega}^i)) \\ &= \sum_{i=1}^{|\mathcal{F}|} m(F^i) \chi_A(\omega^i) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \delta_{\omega^i}(A) \end{aligned}$$

with $\omega^i = X(\overline{\omega}^i) \in \Gamma(\overline{\omega}^i) = F^i$ and $P_{\overline{\Omega}}(\{\overline{\omega}^i\}) = m(F^i)$ where χ_A is the indicator function of A.

So in our above notation the set $\mathcal{M}$ would be generated by

$$\mathcal{M} := \mathcal{M}(\mathcal{F}, m) := \sum_{i=1}^{|\mathcal{F}|} m(F^i)\, \mathcal{M}(F^i)$$

with

$$\mathcal{M}(F^i) = \{\delta_\omega : \delta_\omega(F^i) = 1\} = \{\delta_\omega : \omega \in F^i\} \subseteq \mathcal{K}(F^i) \tag{3}$$

where δ_ω is the Dirac measure at $\omega \in \Omega$ corresponding to the selections. The connections between $\mathcal{M}$ and $\mathcal{K}$ are discussed in [9, 10].

2.3 Sets of parameterized probability measures

Now we generate the set $\mathcal{K}$ of probability measures by a probability measure $p^{\,\theta}$ on $(\Omega, \mathcal{A})$ which is parameterized by an uncertain θ. For modelling the uncertainty of the parameter θ we need the following: A measurable space $(\Theta, \mathfrak{A})$ where Θ is the universal set for θ, $\mathfrak{A}$ a σ-Algebra and $\mathfrak{K}$ a set of probability measures μ on $(\Theta, \mathfrak{A})$. The σ-Algebra $\mathfrak{A}$ has to be chosen in a way that for all $A \in \mathcal{A}$ the mapping

$$\theta \mapsto p^{\,\theta}(A)$$

is $\mathfrak{A}$-measurable.

The set $\mathcal{K}$ is defined by

$$\mathcal{K} := \mathcal{K}(\mathfrak{K}, p^{\,\theta}) := \left\{ P = \int_\Theta p^{\,\theta}(\cdot) \mu(\mathrm{d}\theta) : \; \mu \in \mathfrak{K} \right\}. \tag{4}$$

Then the upper and lower probabilities for a set $A \in \mathcal{A}$ is computed as follows:

$$\overline{P}(A) = \sup\{P(A) : \; P \in \mathcal{K}\} = \sup_{\mu \in \mathfrak{K}} \int_\Theta p^{\,\theta}(A)\; \mu(\mathrm{d}\theta),$$

$$\underline{P}(A) = \inf\{P(A) : \; P \in \mathcal{K}\} = \inf_{\mu \in \mathfrak{K}} \int_\Theta p^{\,\theta}(A)\; \mu(\mathrm{d}\theta).$$

In the following the set $\mathfrak{K}$ is either a set of probability measures generated by ordinary sets or by random sets. The usage and meaning of the symbols $\mathcal{K}$ and $\mathfrak{K}$ is summarized in the following table:

notation	set of probability measures
$\mathcal{K}(F)$	on $(\Omega, \mathcal{A})$ generated by a set F
$\mathfrak{K}(F)$	on $(\Theta, \mathfrak{A})$ generated by a set F
$\mathcal{K}(\mathcal{F}, m)$	on $(\Omega, \mathcal{A})$ gen. by a random set $(\mathcal{F}, m)$
$\mathfrak{K}(\mathcal{F}, m)$	on $(\Theta, \mathfrak{A})$ gen. by a random set $(\mathcal{F}, m)$
$\mathcal{K}(\mathfrak{K}, p^{\,\theta})$	on $(\Omega, \mathcal{A})$ gen. by $\mathfrak{K}$ and $p^{\,\theta}$ as in (4) and where $\mathfrak{K}$ is either a $\mathfrak{K}(F)$ or $\mathfrak{K}(\mathcal{F}, m)$

So $\mathcal{K}$ is always a set of probability measures on $(\Omega, \mathcal{A})$ and $\mathfrak{K}$ a set of probability measures on the parameter space of θ, namely Θ.

2.4 Generation of $\mathfrak{K}$ by probability measures μ on ordinary sets F, $\mathfrak{K} := \mathfrak{K}(F)$

We take the set $\mathfrak{K} := \mathfrak{K}(F)$ of probability measures on $F \in \mathfrak{A}$ and $\mathcal{K} := \mathcal{K}(\mathfrak{K}(F), p^{\,\theta})$ for the set $\mathcal{K}$ of probability measures which are generated by $\mathfrak{K}(F)$ and the parameterized probability measure $p^{\,\theta}$. Then the upper and lower probability are given by

$$\begin{aligned} \overline{P}(A) &= \sup_{\mu \in \mathfrak{K}(F)} \int_\Theta p^{\,\theta}(A)\; \mu(\mathrm{d}\theta) = \qquad (5) \\ &= \sup_{\theta_0 \in F} \int_\Theta p^{\,\theta}(A)\; \delta_{\theta_0}(\mathrm{d}\theta) = \sup_{\theta_0 \in F} p^{\,\theta_0}(A) \end{aligned}$$

and $\underline{P}(A) = \inf_{\theta_0 \in F} p^{\,\theta_0}(A)$. Further we have for the special case $(\Theta, \mathfrak{A}) := (\Omega, \mathcal{A})$ and $p^{\,\omega} := \delta_\omega$:

$$\mathcal{K}(F) = \mathcal{K}(\mathfrak{K}(F), \delta_\omega), \tag{6}$$

because

$$\begin{aligned} \mathcal{K}(\mathfrak{K}(F), \delta_\omega) &= \left\{ P = \int_\Omega \delta_\omega(\cdot) \mu(\mathrm{d}\omega) : \; \mu \in \mathfrak{K}(F) \right\} = \\ &= \{\mu \in \mathfrak{K}(F)\} = \mathfrak{K}(F) = \mathcal{K}(F) \end{aligned}$$

and $\omega \mapsto p^{\,\omega}(A) = \delta_\omega(A) = \chi_A(\omega)$ is $\mathcal{A}$-measurable for all $A \in \mathcal{A}$. So the set of probability measures generated by an ordinary set is integrated into the new concept.

2.5 Generation of $\mathfrak{K}$ by random sets, $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$

Here we take $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$ and $\mathcal{K} := \mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\theta})$. A probability measure $P \in \mathcal{K}$ is written as follows:

$$
\begin{aligned}
P &= \int_{\Theta} p^{\theta}(\cdot)\, \mu(\mathrm{d}\theta) = \\
&= \int_{\Theta} p^{\theta}(\cdot) \left(\sum_{i=1}^{|\mathcal{F}|} m(F^i)\, \mu^i(\mathrm{d}\theta) \right) = \\
&= \sum_{i=1}^{|\mathcal{F}|} m(F^i) \int_{\Theta} p^{\theta}(\cdot)\, \mu^i(\mathrm{d}\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) P^i
\end{aligned}
$$

where $\mu \in \mathfrak{K}(\mathcal{F}, m)$. $\mu = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\mu^i$ is a decomposition of μ according to the focal sets and $P^i = \int_{\Theta} p^{\theta}(\cdot)\mu^i(\mathrm{d}\theta)$ is a probability measure in $\mathcal{K}(\mathfrak{K}(F^i), p^{\theta})$. So for the set $\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\theta})$ we also can write

$$
\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\theta}) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\, \mathcal{K}(\mathfrak{K}(F^i), p^{\theta}) \qquad (7)
$$

which is formula Eq. (2) but with $\mathcal{K}(F^i)$ replaced by $\mathcal{K}(\mathfrak{K}(F^i), p^{\theta})$. The set $\mathcal{K}(F^i)$ used in Eq. (2) is a set of probability measures on F^i, but the probability measures in the set $\mathcal{K}(\mathfrak{K}(F^i), p^{\theta})$ are only associated to F^i via the parameter θ.

Similar to the section above we have for the upper and lower probability:

$$
\begin{aligned}
\overline{P}(A) &= \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\mu^i \in \mathfrak{K}(F^i)} \int_{\Theta} p^{\theta}(A)\mu^i(\mathrm{d}\theta) = \\
&= \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\theta_0 \in F^i} p^{\theta_0}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \overline{P}^i(A)
\end{aligned}
$$

and

$$
\underline{P}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \inf_{\theta_0 \in F^i} p^{\theta_0}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \underline{P}^i(A).
$$

2.6 An example for p^{θ}, gaussian distribution

$(\Omega, \mathcal{A}) := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\Theta, \mathfrak{A}) = (\mathbb{R} \times \mathbb{R}^{>0}, \mathcal{B}(\mathbb{R} \times \mathbb{R}^{>0}))$, $\theta := (\mu, \sigma^2)$. The function

$$
(\mu, \sigma^2) \mapsto p^{(\mu,\sigma^2)}(A) := \int_{\mathbb{R}} \chi_A(x) \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{d}x
$$

is continuous and therefore $\mathfrak{A}$-measurable. We compute the upper and lower probability of the set $A = [a, \infty)$ using a set $\mathfrak{K}(\mathcal{F}, m)$ to describe the uncertainty of μ and σ^2 with $F^i = [\underline{\mu}_i, \overline{\mu}_i] \times [\underline{\sigma}_i^2, \overline{\sigma}_i^2]$, $i = 1, \ldots, n$, as follows:

$$
\overline{P}^i([a,\infty)) = \begin{cases} p^{(\overline{\mu}_i, \underline{\sigma}_i^2)}([a,\infty)) & \overline{\mu}_i \geq a, \\ p^{(\overline{\mu}_i, \overline{\sigma}_i^2)}([a,\infty)) & \text{otherwise,} \end{cases}
$$

and

$$
\underline{P}^i([a,\infty)) = \begin{cases} p^{(\underline{\mu}_i, \underline{\sigma}_i^2)}([a,\infty)) & \underline{\mu}_i \leq a, \\ p^{(\underline{\mu}_i, \overline{\sigma}_i^2)}([a,\infty)) & \text{otherwise.} \end{cases}
$$

Then

$$
\overline{P}([a,\infty)) = \sum_{i=1}^{n} m(F^i) \overline{P}^i([a,\infty))
$$

and

$$
\underline{P}([a,\infty)) = \sum_{i=1}^{n} m(F^i) \underline{P}^i([a,\infty)).
$$

3 Sets of joint probability measures generated by ordinary sets

3.1 Preliminaries

In this paper we restrict ourselves to the combination of only two sets of probability measures. In the following we always need the measurable spaces $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$ and $(\Omega, \mathcal{A})$ with $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ for the uncertain variables and $(\Theta_1, \mathfrak{A}_1)$, $(\Theta_2, \mathfrak{A}_2)$ and $(\Theta, \mathfrak{A})$ with $\Theta = \Theta_1 \times \Theta_2$ for the uncertain parameters of the probabilty measures with σ-Algebras such that mappings like $\theta \mapsto p^{\theta}(A)$ are measurable. A set A will be always in $\mathcal{A}$.

The generation of a set of joint probability measures by two marginal sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of probability measures will be written as $\mathcal{K}(\mathcal{K}_1, \mathcal{K}_2)$. First we recall two general ways of combining sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of probability measures.

Unknown interaction: The set of joint probability measures according to unknown interaction [2] is generated by

$$
\mathcal{K}_{\mathrm{U}} := \{P:\ P(\cdot \times \Omega_2) \in \mathcal{K}_1,\ P(\Omega_1 \times \cdot) \in \mathcal{K}_2\}. \quad \text{(U)}
$$

Strong independence: The set of joint probability measures according to strong independence [2, 11] is generated by

$$
\mathcal{K}_{\mathrm{S}} := \{P_1 \otimes P_2 :\ P_1 \in \mathcal{K}_1,\ P_2 \in \mathcal{K}_2\} \subseteq \mathcal{K}_{\mathrm{U}}. \quad \text{(S)}
$$

Notation for the corresponding probabilities:

$$\begin{aligned}
\overline{P}_{\mathrm{S}}(A) &:= \sup\{P_{\mathrm{S}}(A): \ P_{\mathrm{S}} \in \mathcal{K}_{\mathrm{S}}\},\\
\underline{P}_{\mathrm{S}}(A) &:= \inf\{P_{\mathrm{S}}(A): \ P_{\mathrm{S}} \in \mathcal{K}_{\mathrm{S}}\},\\
\overline{P}_{\mathrm{U}}(A) &:= \sup\{P_{\mathrm{U}}(A): \ P_{\mathrm{U}} \in \mathcal{K}_{\mathrm{U}}\},\\
\underline{P}_{\mathrm{U}}(A) &:= \inf\{P_{\mathrm{U}}(A): \ P_{\mathrm{U}} \in \mathcal{K}_{\mathrm{U}}\}.
\end{aligned}$$

In the following we are analyzing very special cases of sets of joint probability measures which is a preliminary work for Sec. 4 and 5 for dealing with the joint focals sets in these sections.

3.2 $\mathcal{K}_k := \mathcal{K}(F_k) = \mathcal{K}(\mathfrak{K}(F_k), \delta_{\omega_k})$

Given subsets $F_k \in \mathcal{A}_k$, $k = 1, 2$, we generate the sets $\mathcal{K}_{\mathrm{U}}$ and $\mathcal{K}_{\mathrm{S}}$ of joint probability measures by the sets $\mathcal{K}(F_1)$ and $\mathcal{K}(F_2)$. $\mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1), \mathcal{K}(F_2))$ is the set of joint probability measures generated by the two sets $\mathcal{K}(F_1)$ and $\mathcal{K}(F_2)$ of probability measures according to (U). Since the marginals of all probability measures on $F_1 \times F_2$ are in the sets $\mathcal{K}(F_1)$ and $\mathcal{K}(F_2)$, respectively, we have $\mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1), \mathcal{K}(F_2)) = \mathcal{K}(F_1 \times F_2)$. To get the upper and lower probability $\overline{P}_{\mathrm{U}}(A)$ and $\underline{P}_{\mathrm{U}}(A)$ it is sufficient to put a Dirac measure at the appropriate place. Since a Dirac measure is a product measure we get

$$\overline{P}_{\mathrm{S}}(A) = \overline{P}_{\mathrm{U}}(A) \text{ and } \underline{P}_{\mathrm{U}}(A) = \underline{P}_{\mathrm{S}}(A).$$

Now we make a first step towards sets of joint probability measures generated by parameterized probabilities doing the same for $\mathcal{K}\big(\mathfrak{K}(F_k), \delta_{\omega_k}\big)$ in the more general notation. We already know that $\mathcal{K}(F_k) = \mathcal{K}\big(\mathfrak{K}(F_k), \delta_{\omega_k}\big)$ and therefore

$$\begin{aligned}
\mathcal{K}_{\mathrm{U}} &= \mathcal{K}_{\mathrm{U}}\big(\mathcal{K}(F_1), \mathcal{K}(F_2)\big) = \mathcal{K}(F_1 \times F_2) =\\
&= \mathcal{K}\big(\mathfrak{K}(F_1 \times F_2), \delta_{\omega_1} \otimes \delta_{\omega_2}\big).
\end{aligned}$$

Further we have for strong independence

$$\begin{aligned}
\mathcal{K}_{\mathrm{S}} &= \mathcal{K}_{\mathrm{S}}\big(\mathcal{K}(F_1), \mathcal{K}(F_2)\big) =\\
&= \mathcal{K}_{\mathrm{S}}\big(\mathcal{K}(\mathfrak{K}(F_1), \delta_{\omega_1}), \mathcal{K}(\mathfrak{K}(F_2), \delta_{\omega_2})\big) =\\
&= \mathcal{K}\big(\mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2)), \delta_{\omega_1} \otimes \delta_{\omega_2}\big),
\end{aligned}$$

because

$$P_{\mathrm{S}}(A) = \Big(P_1 \otimes P_2\Big)(A) = \int_{\Omega_1} P_2(A_{\omega_1}) P_1(\mathrm{d}\omega_1) = \tag{8}$$

$$\begin{aligned}
&= \int_{\Omega_1} \Big(\int_{\Omega_2}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2)\delta_{\omega_2'}(\mathrm{d}\omega_2)\mu_2(\mathrm{d}\omega_2')\Big) P_1(\mathrm{d}\omega_1) =\\
&= \int_{\Omega_1}\int_{\Omega_1} \Big(\int_{\Omega_2}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2)\delta_{\omega_2'}(\mathrm{d}\omega_2)\mu_2(\mathrm{d}\omega_2')\Big)\cdot\\
&\qquad \cdot \delta_{\omega_1'}(\mathrm{d}\omega_1)\mu_1(\mathrm{d}\omega_1') =\\
&= \int_{\Omega_1}\int_{\Omega_2} \Big(\int_{\Omega_1}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2)\ \delta_{\omega_2'}(\mathrm{d}\omega_2)\delta_{\omega_1'}(\mathrm{d}\omega_1)\Big)\cdot\\
&\qquad \cdot \mu_2(\mathrm{d}\omega_2')\mu_1(\mathrm{d}\omega_1') =\\
&= \int_{\Omega_1}\int_{\Omega_2} \Big[\Big(\delta_{\omega_1'} \otimes \delta_{\omega_2'}\Big)(A)\Big]\ \mu_2(\mathrm{d}\omega_2')\mu_1(\mathrm{d}\omega_1') =\\
&= \int_{\Omega_1 \times \Omega_2} \Big[\Big(\delta_{\omega_1'} \otimes \delta_{\omega_2'}\Big)(A)\Big]\ \mu(\mathrm{d}(\omega_1', \omega_2'))
\end{aligned}$$

with $\mu_1 \in \mathfrak{K}(F_1)$, $\mu_2 \in \mathfrak{K}(F_2)$, $\mu \in \mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2))$ and $A_{\omega_1} = \{\omega_2 \in \Omega_2: \ (\omega_1, \omega_2) \in A\}$.

3.3 $\mathcal{K}_k := \mathcal{K}(\mathfrak{K}(F_k), p_k^{\theta_k})$

Now we replace the Diracs by parameterized probability measures $p_k^{\theta_k}$ and analyze the cases of strong independence and unknown interaction.

3.3.1 Strong independence

Similar to Eq. (8) it holds:

$$\begin{aligned}
P_{\mathrm{S}}(A) &= \int_{\Theta_1}\int_{\Omega_1} \Bigg(\int_{\Theta_2}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2) p_2^{\theta_2}(\mathrm{d}\omega_2)\mu_2(\mathrm{d}\theta_2)\Bigg)\cdot\\
&\qquad \cdot p_1^{\theta_1}(\mathrm{d}\omega_1)\mu_1(\mathrm{d}\theta_1) =\\
&= \int_{\Theta_1}\int_{\Theta_2} \Bigg(\int_{\Omega_1}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2)\ p_2^{\theta_2}(\mathrm{d}\omega_2) p_1^{\theta_1}(\mathrm{d}\omega_1)\Bigg)\cdot\\
&\qquad \cdot \mu_2(\mathrm{d}\theta_2)\mu_1(\mathrm{d}\theta_1) =\\
&= \int_{\Theta_1}\int_{\Theta_2} \Big[\Big(p_1^{\theta_1} \otimes p_2^{\theta_2}\Big)(A)\Big]\ \mu_2(\mathrm{d}\theta_2)\mu_1(\mathrm{d}\theta_1).
\end{aligned}$$

So we get

$$\mathcal{K}_{\mathrm{S}} = \mathcal{K}\big(\mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2)), p_1^{\theta_1} \otimes p_2^{\theta_2}\big).$$

3.3.2 Unknown interaction

For strong independence the joint probability measure generated by $p_1^{\theta_1}$ and $p_2^{\theta_2}$ was $p_1^{\theta_1} \otimes p_2^{\theta_2}$, a single probability measure. In case of unknown interaction we would need the whole set of all possible joint probability measures on $(\Omega, \mathcal{A})$. Maybe on the other hand we have more information how the joint probability measure, say p^θ, is generated by $p_1^{\theta_1}$ and $p_2^{\theta_2}$ than how the parameters of the joint probability measure interact. So we introduce the sets $\mathcal{K}_{(\mathrm{US})}$ and $\mathcal{K}_{(\mathrm{Up}^\theta)}$ of joint probability measures for which the choice of μ is according to (U) and the choice of the joint parameterized probability measure is according to (S) or defined by p^θ.

Then it holds:

$$\begin{aligned}\mathcal{K}_{\mathrm{S}} := \mathcal{K}\big(\mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2)), p_1^{\theta_1} \otimes p_2^{\theta_2}\big) &\subseteq \\ \subseteq \mathcal{K}\big(\mathfrak{K}_{\mathrm{U}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2)), p_1^{\theta_1} \otimes p_2^{\theta_2}\big) &= \\ = \mathcal{K}\big(\mathfrak{K}(F_1 \times F_2), p_1^{\theta_1} \otimes p_2^{\theta_2}\big) &=: \mathcal{K}_{(\mathrm{US})}.\end{aligned}$$

For the upper and lower probabilities we have

$$\overline{P}_{\mathrm{S}}(A) = \overline{P}_{(\mathrm{US})}(A) \text{ und } \underline{P}_{\mathrm{S}}(A) = \underline{P}_{(\mathrm{US})}(A),$$

because we can obtain the upper and lower probabilities from $\mathcal{K}_{(\mathrm{US})}$ by means of Dirac measures in $\mathfrak{K}(F_1 \times F_2)$ which are also in $\mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(F_1), \mathfrak{K}(F_2))$.

4 General formulation of the generation of sets of joint probability measures by random sets

Let random sets $(\mathcal{F}_k, m_k)$, $k = 1, 2$, be given for modelling the uncertainty of the variables x_1 and x_2. As a consequence of Dempster's rule of combination [3, 4] the joint random set $(\mathcal{F}, m)$ is defined by

$$\mathcal{F} = \{F^{ij} : \ i = 1, \ldots, n_1; \ j = 1, \ldots, n_2\}$$

where

$$F^{ij} := F_1^i \times F_2^j$$

and

$$m(F_1^i \times F_2^j) := m_1(F_1^i) m_2(F_2^j) \tag{9}$$

which is the case of random set independence (RS-independence).

For our more general approach we start with the multivariate analogon of Eq. (2):

$$\mathcal{K}_? = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) \mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

where the question mark in $\mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$ indicates the possibility of different choices in combining the sets of probability measures $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ associated with the marginal focal sets F_1^i and F_2^j. Further we have to define how the joint weights $m(F_1^i \times F_2^j)$ are computed (perhaps not in the way of Eq. (9)) and to think about possible interactions between probability measures in the different sets $\mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$.

The consequences of these different choices are different sets of joint probability measures $\mathcal{K}_?$ and the goal is to generate sets according to strong independence, unknown interaction and RS-independence. In the following we describe the different choices we have for the above formula and discuss their consequences for the set of joint probability measures.

4.1 The choice of the joint weights $m(F_1^i \times F_2^j)$

The weights m_1 and m_2 are discrete probability measures on the sets of focal sets $\{F_1^1, \ldots, F_1^{n_1}\}$, $\{F_2^1, \ldots, F_2^{n_2}\}$ respectively. So if we want to choose the joint focal sets in a stochastically independent way, then $m = m_1 \otimes m_2$ which means $m(F_1^i \times F_2^j) = m_1(F_1^i) m_2(F_2^j)$ for all i, j. If we do not know how m_1 and m_2 interact, we allow all possible combinations, that means unknown interaction.

Case (U−−): Unknown interaction, m must satisfy the following conditions:

$$\begin{aligned} m_1(F_1^i) &= \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j), \quad i = 1, \ldots, |\mathcal{F}_1|, \\ m_2(F_2^j) &= \sum_{i=1}^{|\mathcal{F}_1|} m(F_1^i \times F_2^j), \quad j = 1, \ldots, |\mathcal{F}_2|. \end{aligned}$$

In this case m is not uniquely defined and is determined later on by solving an optimization problem for the lower or upper probabilities.

Case (S−−): Stochastic independence:

$$m(F_1^i \times F_2^j) := m_1(F_1^i) m_2(F_2^j).$$

4.2 The choice of P^{ij}, $\mathcal{K}^{ij}$, respectively

$P^{ij} \in \mathcal{K}^{ij}$ is a probability measure associated to the joint focal set $F_1^i \times F_2^j$. How a P^{ij} looks like depends on how $\mathcal{K}^{ij}$ is constructed from $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$.

Case (−U−): $\mathcal{K}_{\mathrm{U}}^{ij} := \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j))$ which is the set of all joint probability measures generated by the sets $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ according to condition (U).

Case (−S−): $\mathcal{K}_{\mathrm{S}}^{ij} := \mathcal{K}_{\mathrm{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$ which is the set generated according to strong independence (S).

4.3 The choice of interactions between the P^{ij}

Case (−−1): Row- and columnwise equality conditions on the marginals of the probability measures on the joint focal sets:

$$\begin{aligned} P_1^i &:= P_1^{i,i1} = \cdots = P_i^{i,in_2}, \ i = 1, \ldots, n_1, \\ P_2^j &:= P_2^{j,1j} = \cdots = P_i^{j,n_1 j}, \ j = 1, \ldots, n_2 \end{aligned}$$

where

$$P_1^{i,ik} = P_1^{ik}(\cdot \times \Omega_2) \text{ and } P_2^{j,kj} = P_2^{kj}(\Omega_1 \times \cdot).$$

This condition seems to be very artificial, but we need this to get results according to strong independence later on.

Case (−−0): No interactions, this means that we can choose a $P^{ij} \in \mathcal{K}^{ij}$ on $F_1^i \times F_2^j$ irrespective of the probability measures chosen on other joint focal sets.

Remark: It is clear that it should hold that the convex sum

$$\sum_k \frac{1}{m_1(F_1^i)} m(F_1^i \times F_2^j) P_1^{i,ik}$$

is in $\mathcal{K}_1^i$. This is always true for convex sets $\mathcal{K}_1^i$ of probability measures, but for sets which are generated by measurable selections (see Eq. (3)) it is not true in general. In this case one should introduce a more restrictive condition than (−−1).

4.4 The choice of the joint marginals

We emphasize that the choice of the Cartesian products $F_1^i \times F_2^j$ as joint focals is no restriction of generality. Joint focal sets $V \subseteq F_1^i \times F_2^j$ of arbitrary shape can be subsumed in our approach by restricting sets of joint probability measures on $F_1^i \times F_2^j$ to those whose support lies in V. Such subsets would describe specific types of dependence or interaction between the marginal focal sets F_1^i and F_2^j. But such interactions are not investigated in this paper.

5 The different cases

Now we will discuss combinations of the above cases which lead to random set independence, unknown interaction, strong independence. The cases are indicated by indices of the form (ABC) where for example (SU0) means m according (S−−), P^{ij} according to (−U−) and no interaction between the P^{ij}.

We want to stress that again, that it is not our goal to introduce a number of eight (all possible combinations) new types of joint probability measures, but to identify the combinations which leads to the desired types of sets joint probability measures. We do this for RS-independence, unknown interaction and strong independence. In this very technical part we first recall for each of these types the case where "pure random sets" are used, that means the case where no parameterized probabilities are involved. Then we generalize the results to the case of parameterized probabilities. So the sets $\mathcal{K}_k^i$ are first sets of probability measures $\mathcal{K}(F_k^i)$ and then in a second part replaced by sets $\mathcal{K}(\mathfrak{K}(F_k^i), p^{\theta})$ associated with the marginal focal set F_k^i.

5.1 (SU0), (SS0) and RS-independence

5.1.1 General formulation

The sets $\mathcal{K}_{\mathrm{SU0}}$ and $\mathcal{K}_{\mathrm{SS0}}$ of joint probability measures are generated by

$$\mathcal{K}_{\mathrm{SU0}} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

$$\mathcal{K}_{\mathrm{SS0}} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j) m_2(F_2^j) \mathcal{K}_{\mathrm{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j).$$

5.1.2 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

We obtain the upper probability $\overline{P}_{\mathrm{SU0}}(A)$ for a set $A \in \mathcal{A}$ by

$$\overline{P}_{\mathrm{SU0}}(A) = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \overline{P}_{\mathrm{U}}^{ij}(A)$$

where

$$\overline{P}_{\mathrm{U}}^{ij}(A) = \sup \left\{ P_{\mathrm{U}}^{ij}(A) : \ P_{\mathrm{U}}^{ij} \in \mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1^i), \mathcal{K}(F_2^j)) \right\}$$

and

$$\mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1^i), \mathcal{K}(F_2^j)) = \mathcal{K}(F_1^i \times F_2^j).$$

So $\overline{P}_{\mathrm{U}}^{ij}(A)$ is computed very easily by

$$\overline{P}_{\mathrm{U}}^{ij}(A) = \sup\{\delta_\omega(A) : \ \omega \in F_1^i \times F_2^j\} = \\ = \begin{cases} 1 & \exists\, \omega \in A \cap F_1^i \times F_2^j, \\ 0 & \text{else} \end{cases}$$

which leads to the formula for the joint plausibility measure

$$\overline{P}_{\mathrm{R}}(A) := P_{\mathrm{SU0}}(A) = \mathrm{Pl}(A) = \sum_{i,j:\ F_1^i \times F_2^j \cap A \neq \varnothing} m_1(F_1^i) m_2(F_2^j)$$

which is the joint upper probability in the case of RS-independence indicated by the index R. Further we have $\overline{P}_{\mathrm{SU0}} = \overline{P}_{\mathrm{SS0}}$ because

$$\delta_\omega = \delta_{(\omega_1,\omega_2)} = \delta_{\omega_1} \otimes \delta_{\omega_2}.$$

is a product measure (case (−S−)). Similar to the upper probability we get for the lower probability

$$\underline{P}_{\mathrm{R}} := \mathrm{Bel} = \underline{P}_{\mathrm{SU0}} = \underline{P}_{\mathrm{SS0}}.$$

Contrary to the above equalities we have for the corresponding sets of joint probability measures only

$$\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{SU0}} \supseteq \mathcal{K}_{\mathrm{SS0}}.$$

5.1.3 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

An idea would be to define $\mathcal{K}_\mathrm{R}$ by $\mathcal{K}_\mathrm{SU0}$ as before [6], but then we have the same problem as in Sec. 3.3.2. So another possibility would be to define $\mathcal{K}_\mathrm{R} := \mathcal{K}_\mathrm{S(US)0}$ or $\mathcal{K}_\mathrm{R} := \mathcal{K}_{\mathrm{S(Up^\theta)0}}$.

We start with the case of (SS0) and get

$$\begin{aligned}\mathcal{K}_\mathrm{SS0} &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\,\mathcal{K}_\mathrm{S}^{ij}\\ &\subseteq \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\mathcal{K}_\mathrm{(US)}^{ij} = \\ &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\cdot \\ &\qquad \cdot\,\mathcal{K}\big(\mathfrak{K}(F_1^i\times F_2^j), p_1^{\theta_1}\otimes p_2^{\theta_2}\big) = \\ &= \mathcal{K}\big(\mathfrak{K}(\mathcal{F}, m), p_1^{\theta_1}\otimes p_2^{\theta_2}\big) =: \mathcal{K}_\mathrm{S(US)0} =: \mathcal{K}_\mathrm{R},\end{aligned}$$

with $\mathcal{K}_\mathrm{S}^{ij} := \mathcal{K}_\mathrm{S}\big(\mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1}), \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})\big)$ and $\mathcal{K}_\mathrm{(US)}^{ij} := \mathcal{K}\big(\mathfrak{K}(F_1^i\times F_2^j), p_1^{\theta_1}\otimes p_2^{\theta_2}\big)$ and Eq. (7). $(\mathcal{F}, m)$ is the joint random set according to RS-independence. $\mathcal{K}_\mathrm{S(SU)0}$ is the set of probability measures where the parameterized probability measure is the product measure, but the uncertainty of the parameters of this poduct measure is discribed by the set $\mathfrak{K}(\mathcal{F}, m)$ of joint probability measures which are generated by the random set describing the uncertainty of θ_1 and θ_2.

For the upper and lower probabilities we have $\overline{P}_\mathrm{SS0} = \overline{P}_\mathrm{S(US)0}$ and $\underline{P}_\mathrm{SS0} = \underline{P}_\mathrm{S(US)0}$ by the same arguments as in Sec. 3.3.2.

5.2 (UU0), (US0) and unknown interaction

5.2.1 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

Let $\mathcal{K}_\mathrm{UU0}$ be the set of probability measures generated according to case (UU0). A computational method for $\overline{P}_\mathrm{UU0}(A)$ is obtained in the following way:

$$\begin{aligned}\overline{P}_\mathrm{UU0}(A) &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m^*(F_1^i\times F_2^j)\,\overline{P}_\mathrm{U}^{ij}(A) = \\ &= \sum_{\substack{i,j:\\ F_1^i\times F_2^j\cap A\neq\varnothing}} m^*(F_1^i\times F_2^j),\end{aligned}$$

where $\overline{P}_\mathrm{U}^{ij}(A)$ is computed by the same Dirac measures as for $\overline{P}_\mathrm{R}$ and the weights m^* by solving the following linear optimization problem:

$$\sum_{\substack{i,j:\\ F_1^i\times F_2^j\cap A\neq\varnothing}} m(F_1^i\times F_2^j) = \max!$$

subject to condition (U−−). Minimization instead of maximization leads to lower probability $\underline{P}_\mathrm{UU0}(A)$.

The set $\mathcal{K}_\mathrm{UU0}$ is just the set of probability measures which is generated by the least restrictive conditions on m and P^{ij}. It is proven in [5, 6] that $\mathcal{K}_\mathrm{U} = \mathcal{K}_\mathrm{UU0}$.

By the same arguments as in the previous cases we get $\overline{P}_\mathrm{U} = \overline{P}_\mathrm{UU0} = \overline{P}_\mathrm{US0}$ and $\underline{P}_\mathrm{U} = \underline{P}_\mathrm{UU0} = \underline{P}_\mathrm{US0}$.

5.2.2 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

Similar to Sec. 5.1.3 we can define sets

$$\begin{aligned}\mathcal{K}_\mathrm{UU0} &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i\times F_2^j)\cdot \\ &\qquad \cdot\mathcal{K}_\mathrm{U}\big(\mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1}), \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})\big)\\ \mathcal{K}_\mathrm{US0} &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i\times F_2^j)\cdot \\ &\qquad \cdot\mathcal{K}\big(\mathfrak{K}_\mathrm{S}(\mathfrak{K}(F_1^i), \mathfrak{K}(F_2^j)), p_1^{\theta_1}\otimes p_2^{\theta_2}\big)\\ \mathcal{K}_\mathrm{U(US)0} &= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i\times F_2^j)\cdot \\ &\qquad \cdot\mathcal{K}\big(\mathfrak{K}(F_1^i\times F_2^j), p_1^{\theta_1}\otimes p_2^{\theta_2}\big),\end{aligned}$$

where in addition the joint weights can be choosen according to (U−−); and it also holds
$\overline{P}_\mathrm{US0}(A) = \overline{P}_\mathrm{U(US)0}$, $\underline{P}_\mathrm{US0}(A) = \underline{P}_\mathrm{U(US)0}$ and
$\mathcal{K}_\mathrm{UU0} \supseteq \mathcal{K}_\mathrm{U(US)0} \supseteq \mathcal{K}_\mathrm{US0}$.

But unfortunately we do not have $\mathcal{K}_\mathrm{U} = \mathcal{K}_\mathrm{UU0}$ in general what we show in the following example.

Example:

The sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of probability measures are given by

$\mathcal{K}_1 = \mathcal{K}(\mathcal{F}_1, m_1) = \mathcal{K}(\mathfrak{K}(\mathcal{F}_1, m_1), \delta_\omega)$

and

$\mathcal{K}_2 = \mathcal{K}(\mathfrak{K}(\mathcal{F}_2, m_2), p_2^{\theta_2})$

where $p_2^{\theta_2}$ is defined by $p_2^{\theta_2}(\{0\}) = \theta_2$ and
$p_2^{\theta_2}(\{1\}) = 1-\theta_2$ and where

$\Omega_1 = \Omega_2 = \{0,1\}$, $\Omega = \Omega_1\times\Omega_2$
$\mathcal{F}_1 = \{\{0\},\{1\}\}$, $m_1(\{0\}) = m_1(\{1\}) = \frac{1}{2}$,
$\mathcal{F}_2 = \{\{\frac{1}{2}\}\}$, $m_2(\{\frac{1}{2}\}) = 1$.

In this very special example both marginal sets of probability measures have only one element, namely the discrete uniform distribution on $\{0,1\}$:

$\mathcal{K}_1 = \{P_1^1\}$, $\mathcal{K}_2 = \{P_2^1\}$, $P_1 = P_2$ and

$P_1(\{0\}) = P_1(\{1\}) = \frac{1}{2}$.

But this uniform distribution is “generated” by two different ways:

1. As a degenerated random set where the two focal sets are singletons.

2. As a realization of the parameterized probability measure $p_2^{\theta_2}$ with a parameterization by a random set with only one focal set.

The sets of probability measures associated with the marginal focal sets are given by

$\mathcal{K}_1^1 = \{P_1^1\},\ P_1^1(\{0\}) = 1,$

$\mathcal{K}_1^2 = \{P_1^2\},\ P_1^2(\{1\}) = 1,$

$\mathcal{K}_2^1 = \{P_2^1\} = \{P_2\}.$

Now we determine the joint focal sets and weights:

$\mathcal{F} = \{F^{11}, F^{21}\}$ with $F^{11} = \{(0, \frac{1}{2})\}$, $F^{21} = \{(1, \frac{1}{2})\}$,

$m(F^{11}) = m(F^{21}) = \frac{1}{2}$.

Since $|\mathcal{F}_2| = 1$ the joint weights are uniquely determined independent of (S−−) or (U−−).

The sets of probability measures associated with the joint focal sets:

$\mathcal{K}_{\mathrm{U}}^{11} = \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^1, \mathcal{K}_2^1) = \mathcal{K}_{\mathrm{U}}(P_1^1, P_2^1) = \{P_{\mathrm{U}}^{11}\}$ with

$P_{\mathrm{U}}^{11}(\{(0,0)\}) = P_{\mathrm{U}}^{11}(\{(0,1)\}) = \frac{1}{2}$

and

$\mathcal{K}_{\mathrm{U}}^{21} = \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^2, \mathcal{K}_2^1) = \mathcal{K}_{\mathrm{U}}(P_1^2, P_2^1) = \{P_{\mathrm{U}}^{21}\}$ with

$P_{\mathrm{U}}^{21}(\{(1,0)\}) = P_{\mathrm{U}}^{21}(\{(1,1)\}) = \frac{1}{2}$.

Let $A = \{(0,0), (1,1)\}$. Then

$\overline{P}_{\mathrm{UU0}}(A) = m(F^{11})P_{\mathrm{U}}^{11}(A) + m(F^{21})P_{\mathrm{U}}^{21}(A) =$

$= \frac{1}{2}P_{\mathrm{U}}^{11}(\{(0,0)\}) + \frac{1}{2}P_{\mathrm{U}}^{21}(\{(1,1)\}) =$

$= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$.

But it is clear that

$\overline{P}_{\mathrm{U}}(A) = \sup\{P_{\mathrm{U}}(A) : \ P_{\mathrm{U}} \in \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1, \mathcal{K}_2)\} = 1$ for P_{U} defined by $P_{\mathrm{U}}(\{(0,0)\}) = P_{\mathrm{U}}(\{(1,1)\}) = \frac{1}{2}$.

5.3 The case (SS1), strong independence

5.3.1 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

We write a probability measure $P_{\mathrm{SS1}} \in \mathcal{K}_{\mathrm{SS1}}$ in the following way:

$$P_{\mathrm{SS1}}(A) = \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\,(P_1^i \otimes P_2^j)(A) =$$
$$= \left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i)\,P_1^i\right) \otimes \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j)\,P_2^j\right)(A) =$$
$$= (P_1 \otimes P_2)(A) = P_{\mathrm{S}}(A)$$

with $P_1 \in \mathcal{K}(\mathcal{F}_1, m_1)$ and $P_2 \in \mathcal{K}(\mathcal{F}_2, m_2)$. This leads to

$$\mathcal{K}_{\mathrm{SS1}} = \mathcal{K}_{\mathrm{S}} =$$
$$= \big\{P_1 \otimes P_2 : P_1 \in \mathcal{K}(\mathcal{F}_1, m_1), P_2 \in \mathcal{K}(\mathcal{F}_2, m_2)\big\}$$

which is the case of strong independence.

Computational method:

Theorem 1. *The upper probability $\overline{P}_{\mathrm{S}}(A)$ is the solution of the following global optimization problem:*

$$\sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j)\,\chi_A(\omega_1^i, \omega_2^j) = \max!$$

subject to

$$\omega_1^i \in F_1^i,\ i = 1, \ldots, |\mathcal{F}_1|,$$
$$\omega_2^j \in F_2^j,\ j = 1, \ldots, |\mathcal{F}_2|,$$

where χ_A is the indicator function of the set A. The lower probability $\underline{P}_{\mathrm{S}}(A)$ is obtained by minimization.

Proof: see [5, 8].

In general it is very hard to solve the above optimization problem because there may be many local maxima (or minima) and because the objective function is not continuous. Criteria when we have $\overline{P}_{\mathrm{S}} = \overline{P}_{\mathrm{R}}$ are given in [6]. In this case we automatically get $\overline{P}_{\mathrm{S}}$ by using the computationally cheaper $\overline{P}_{\mathrm{R}}$.

5.3.2 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

It holds

$$P_{\mathrm{S}}(A) = P_{\mathrm{SS1}}(A) = \big(P_1 \otimes P_2\big)(A) =$$
$$= \left[\left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i)P_1^i\right) \otimes \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j)P_2^j\right)\right](A) =$$
$$= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\Big(P_1^i \otimes P_2^j\Big)(A) =$$
$$= \sum_{i=1}^{|\mathcal{F}_1|}\sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i)m_2(F_2^j)\cdot$$
$$\cdot \int_{\Theta_1}\int_{\Theta_2}\left(\int_{\Omega_1}\int_{\Omega_2} \chi_{A_{\omega_1}}(\omega_2)\ p_2^{\theta_2}(\mathrm{d}\omega_2)p_1^{\theta_1}(\mathrm{d}\omega_1)\right)\cdot$$
$$\cdot\ \mu_2^j(\mathrm{d}\theta_2)\mu_1^i(\mathrm{d}\theta_1) =$$

$$
\begin{aligned}
&= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \cdot \\
&\quad \cdot \int_{\Theta_1} \int_{\Theta_2} \left[\left(p_1^{\theta_1} \otimes p_2^{\theta_2} \right)(A) \right] \mu_2^j(\mathrm{d}\theta_2) \mu_1^i(\mathrm{d}\theta_1) = \\
&= \int_{\Theta_1} \int_{\Theta_2} \left[\left(p_1^{\theta_1} \otimes p_2^{\theta_2} \right)(A) \right] \mu_2(\mathrm{d}\theta_2) \mu_1(\mathrm{d}\theta_1) = \\
&= \int_{\Theta_1 \times \Theta_2} \left[\left(p_1^{\theta_1} \otimes p_2^{\theta_2} \right)(A) \right] \mu(\mathrm{d}(\theta_1, \theta_2))
\end{aligned}
$$

with $\mu \in \mathfrak{K}_{\mathrm{S}}(\mathfrak{K}(\mathcal{F}_1, m_1), \mathfrak{K}(\mathcal{F}_2, m_2))$, $\mu_1^i \in \mathfrak{K}(F_1^i)$, $\mu_2^j \in \mathfrak{K}(F_2^j)$, $\mu_1 \in \mathfrak{K}(\mathcal{F}_1, m_1)$ and $\mu_2 \in \mathfrak{K}(\mathcal{F}_2, m_2)$.

Computational method:

We get the following optimization problem for the computation of $\overline{P}_{\mathrm{S}}(A)$ and $\underline{P}_{\mathrm{S}}(A)$, respectively:

$$
\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \left(p_1^{\theta_1^i} \otimes p_2^{\theta_2^j} \right)(A) = \sup! \quad (\inf!)
$$

subject to $\theta_1^i \in F_1^i$ and $\theta_2^j \in F_2^j$. Proof: see [6].

6 Summary

We summarize the results where parameterized probabilities are involved: Fig. 1 depicts the relations between the sets of joint probability measures. For the upper probabilities see Fig. 2. There are three differences to the results for "pure random sets" in [5].

1. $\mathcal{K}_{\mathrm{UU0}}$ is only a subset of $\mathcal{K}_{\mathrm{U}}$ in general.
2. New cases induced by $(-(\mathrm{US})-)$ which coincide with $(-\mathrm{U}-)$ for "pure random sets" because of the Dirac measures.
3. Generalization of the computational method for $\overline{P}_{\mathrm{S}}$ and $\underline{P}_{\mathrm{S}}$.

$$
\begin{array}{ccccc}
\boxed{\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{K}_{\mathrm{UU0}}} & \supseteq & \mathcal{K}_{\mathrm{U(US)0}} & \supseteq & \mathcal{K}_{\mathrm{US0}} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathcal{K}_{\mathrm{SU0}} & \supseteq & \boxed{\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{S(US)0}}} & \supseteq & \mathcal{K}_{\mathrm{SS0}} \\
 & & & & \rotatebox{90}{$\subseteq$} \\
 & & & & \boxed{\mathcal{K}_{\mathrm{S}} = \mathcal{K}_{\mathrm{SS1}}}
\end{array}
$$

Figure 1: Relations between the sets of probability measures.

$$
\begin{array}{ccccc}
\boxed{\overline{P}_{\mathrm{U}} \geq \overline{P}_{\mathrm{UU0}}} & \geq & \overline{P}_{\mathrm{U(US)0}} & = & \overline{P}_{\mathrm{US0}} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\overline{P}_{\mathrm{SU0}} & \geq & \boxed{\overline{P}_{\mathrm{R}} := \overline{P}_{\mathrm{S(US)0}}} & = & \overline{P}_{\mathrm{SS0}} \\
 & & & & \rotatebox{90}{$\leq$} \\
 & & & & \boxed{\overline{P}_{\mathrm{S}} = \overline{P}_{\mathrm{SS1}}}
\end{array}
$$

Figure 2: Relations between the upper probabilities.

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