# *Decision Theories for Some IPs – what dominance gains for us*

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## Outline

1. **Review of structural (not normative) assumptions for this overview.**
   - **Moral hazards and dominance**
   - **State-dependent utilities**
   - **Normal versus extensive form decision making**

2. **IP theory relating to de Finetti's *coherence of previsions*.**
   - ***Fundamental Theorem*: *imprecise* vs. *indeterminate* previsions**
   - **Limitations of binary comparisons with *coherent* IP-sets**
   - **Dominance and coherent previsions for unbounded variables**

3. **IP theories related to relaxing canonical axioms of SEU theory**
   - **$\Gamma$-*Maximin*, *Maximality*, and *E-Admissibility***
   - **On dominance and rationalizability-- D.Pearce's (1984) result.**

## 1. Structural assumptions used in this overview

- ***Act-state independence*: no cases of "moral hazards" are considered – so strict dominance is valid.**

***Reminder:*** **Consider the following binary state, two act decision problem, with outcomes ordinally (or cardinally) ranked so that more is better.**

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| **$Act_1$** | **3** | **1** |
| **$Act_2$** | **4** | **2** |

**$Act_2$ strictly dominates $Act_1$. Nonetheless, if**

$$Prob(\omega_i \mid Act_i) \approx 1 \quad (i = 1, 2),$$

**then dominance carries no force. A rational decision maker prefers $Act_1$ to $Act_2$.**

**In today's overview only decision problems <u>without</u> *moral hazards* are considered.**

- *State-independent utility*: **no cases where the value of a prize depends upon the state in which it is received.**

***Reminder:*** **Once we entertain, generalized state-dependent utilities for prizes, there is maximal under-determination (= up to *mutual absolute continuity* of probability) of probability/utility pairs that represent the very same preference ranking of acts.**

## Matrix of *m*-many acts on the partition of *n*-many uncertain states

| | $\omega_1$ | $\omega_2$ | | $\omega_j$ | | | $\omega_n$ |
|---|---|---|---|---|---|---|---|
| $Act_1$ | $o_{11}$ | $o_{12}$ | | $o_{1j}$ | | | $o_{1n}$ |
| $Act_2$ | $o_{21}$ | $o_{22}$ | | $o_{2i}$ | | | $o_{2n}$ |
| | | | | | | | |
| $Act_i$ | $o_{i1}$ | $o_{i2}$ | | $o_{ij}$ | | | $o_{in}$ |
| | | | | | | | |
| $Act_m$ | $o_{m1}$ | $o_{m2}$ | | $o_{mj}$ | | | $o_{mn}$ |

**In accord with generalized SEU preference over acts, suppose**

**$Act_1$ is dispreferred to $Act_2$ *if and only if* $\sum_j P(\omega_j)U_j(o_{1j}) \leq \sum_j P(\omega_j)U_j(o_{2j})$**

**Choose P* mutually absolutely continuous with P and define the constants**

$$c_j = P(\omega_j)/P^*(\omega_j)$$

**and let** $$U^*_j(\bullet) = c_j U_j(\bullet) \qquad (j = 1, \ldots, n).$$

**Then, trivially,**

$$\sum_j P(\omega_j)U_j(o_{1j}) \leq \sum_j P(\omega_j)U_j(o_{2j}) \text{ if and only if } \sum_j P^*(\omega_j)U^*_j(o_{1j}) \leq \sum_j P^*(\omega_j)U^*_j(o_{2j}).$$

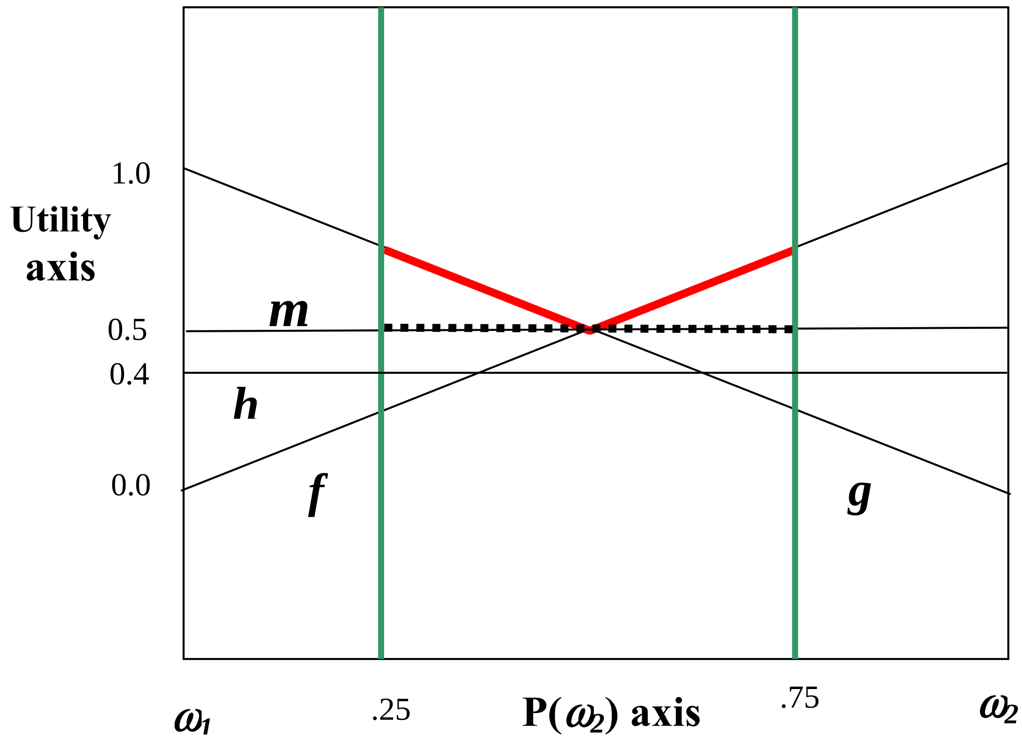
**Note well: This problem arises even when one of the representations is by a state-independent utility!**

- **The focus here is on *normal* form (aka "strategic form") games/decisions. This is where the decision maker can commit, in advance, to all contingency planning.**

***Reminder*: We make no assumption that *normal* and *extensive* form decisions are equivalent, and generally they will not be equivalent for IP-decision theories.**

- **This issue is particularly important for so-called "dynamic Book" arguments**

***Dilation Example* (in the spirit of Ellsberg's Paradox)**



**Consider the following *normal* and *extensive* decision problems.**

***Normal* (strategic) form:**

- **Choose among the mixed option *m* and constant acts *h* and *0*.**

  ***m* is defined as *f* if *heads* and *g* if *tails* on a flip of a *fair* coin.**

- **All (?) decision rules recommend *m* uniquely from the trio {0, *h*, *m*} in the normal form**

***Extensive* (sequential) form:**

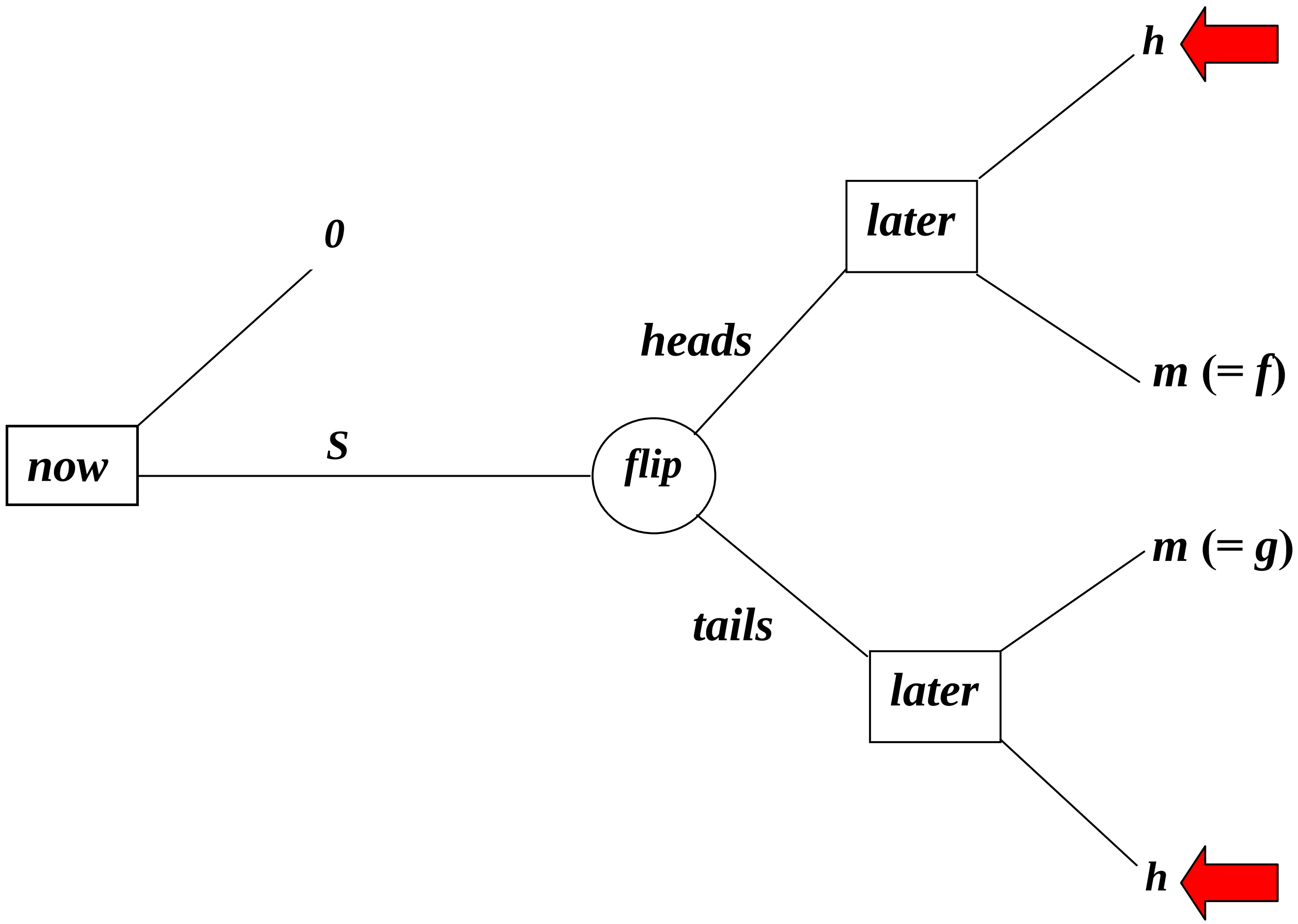
- **Choose *now* between the constant act *0* and the sequential option *S***

  **With *S* you observe the fair coin flip and *later* choose between *m* and *h*.**

  **But, later *m* is equivalent to *f* if heads, or equivalent to *g* if tails.**

  **Many IP-decision rules will advise *h* in a pairwise choice either between *h* and *f*, or between *h* and *g*.**

  - **Then, *h* (not *m*) is the outcome of the *extensive* form decision.**


h
later
0
heads
m (= f)
now
S
flip
m (= g)
tails
later
h

## 2. IP theory relating to de Finetti's *coherence of previsions*.

**An important, and historically early application of strict dominance for IP-decision theory is based on de Finetti's criterion of *coherence of previsions*. Let us first review de Finetti's two related theorems.**

***Coherence* – de Finetti's notion of coherence begins with**

**an arbitrary partition of states, $\Omega = \{\omega_i: i \in I\}$, and**

**an arbitrary collection of real random variables, $\chi = \{X_j: j \in J\}$, defined on $\Omega$.**

**For each random variable $X \in \chi$, the rational agent has a (two-sided) *prevision* P(X) which is to be interpreted as a *fair* price (both for *buying* and *selling*) as follows.**

**For all real $\beta > 0$, small enough so that the agent is willing to pay the possible losses,**

**the agent is willing to pay $\beta P(X)$ in order *to buy* (i.e., to receive) $\beta X$ in return.**

**and, willing to accept $\beta P(X)$ in order *to sell* (i.e., to pay) $\beta X$ in return.**

**In symbols, the agent will accept the gamble**

$$\beta[X - P(X)]$$

**as a change in fortune, for all sufficiently small (positive or negative) $\beta$.**

**The agent is required to accept all finite sums of gambles of the preceding form.**

**That is, for all finite *n* and all small, real $\beta_1, \ldots, \beta_n$ and all $X_1, \ldots, X_n \in \chi$, the agent will accept the combination of gambles**

$$\sum_{i=1}^{n} \beta_i[X_i - P(X_i)].$$

**Where $\beta_i$ is positive, the agent buys $\beta_i$-units of $X_i$ for a price of $\beta_i P(X_i)$**

**where it is negative, the agent sells $\beta_i$-units of $X_i$ for a price of $\beta_i P(X_i)$.**

**The previsions are *incoherent* if there is a *uniformly negative* acceptable finite combination of gambles. That is, if there exists a sum of the form above and $\varepsilon > 0$ such that, for each $\omega \in \Omega$,**

$$\sum_{i=1}^{n} \beta_i[X_i(\omega) - P(X_i)] < -\varepsilon.$$

**Otherwise the agent's previsions are *coherent.***

**Where previsions are incoherent, the book that indicates this constitutes a combination of gambles that is uniformly, strictly dominated by *not-betting* (= 0).**

**de Finetti's *Coherence Theorem*:**

- **A set of previsions are coherent if and only if they are the expected values for the respective random variables under a (finitely additive) probability distribution over Ω.**

- **When the variables are indicator functions for events (subsets of Ω), coherent previsions are exactly those in agreement with a (finitely additive) probability. And then the $| \beta_i |$ are the stakes in winner-take-all bets, where the previsions fix betting rates, $P(X_i) : 1-P(X_i)$.**

- **de Finetti's *Fundamental Theorem of Previsions***

  **Suppose coherent (2-sided) previsions are given for all variables in a set $\chi$ defined with respect to $\Omega$.**

**Let Y be a real-valued function defined on $\Omega$ but not in $\chi$.**

**Define:** $\underline{A} = \{X: X(\omega) \leq Y(\omega)$ **and X is in the linear span of** $\chi\}$

$\overline{A} = \{X: X(\omega) \geq Y(\omega)$ **and X is in the linear span of** $\chi\}$

**Let**

$$\underline{P}(Y) = \sup_{X \in \underline{A}} P(X) \quad \text{and} \quad \overline{P}(Y) = \inf_{X \in \overline{A}} P(X)$$

**<u>Then</u> the 2-sided prevision, P(Y), may be any finite number from $\underline{P}(Y)$ to $\overline{P}(Y)$ and the resulting enlarged set of previsions is coherent.**

**Outside this interval, the enlarged set of previsions is incoherent.**

**The *Fundamental Theorem* provides an early instance of IP-theory where, in I.Levi's (1980) terms relating to I.J.Good's "Black Box Theory, the 'I' stands for**

**an *imprecise* prevision, rather than an *indeterminate* prevision.**

**That is,**

> **The interval for a new prevision [$\underline{P}$ (Y) $\overline{P}$ (Y)] given by the *Fundamental Theorem* constrains a new, 2-sided prevision for a variable, $Y \notin \chi$, while preserving coherence of the 2-sided previsions already assigned to $X \in \chi$.**

**Thus, coherence for 2-sided previsions does not require the rational agent to identify precise previsions beyond those in the linear span of the variables in the arbitrary set $\chi$.**

**Specifically, the rational agent is not required by *coherence* to have determinate probabilities defined on an algebra of events, let alone on a power-set of events. It is sufficient to have probabilities defined *as-needed* for the arbitrary set $\chi$, as might arise in a particular decision problem.**

- **See, e.g., F. Lad, 1996 for interesting applications of this result.**

- ***Toy Example 1.1*** **– good, realistic examples are given by Lad:**

  **$\Omega = \{1, 2, 3, 4, 5, 6\}$ the outcome of rolling an ordinary die.**

  **$\chi$ is the set of indicator functions for the following four events**

$$\chi = \{ \{1\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \}$$

**Suppose 2-sided previsions for these four events are given, and agree with the judgment that the die is "fair."**

$$P(\{1\}) = 1/6; \quad P(\{3,6\}) = 1/3; \quad P(\{1,2,3\}) = P(\{1,2,4\}) = 1/2.$$

**The set of events for which a precise prevision is fixed by the 2-sided previsions for these four events is given by the *Fundamental Theorem*.**

- **That set does *<u>not</u>* form an algebra. Only 22 of 64 events have precise previsions.**

  **For instance, by the Fundamental Theorem,**

$$\underline{P}(\{6\}) = 0 < \overline{P}(\{6\}) = 1/3;$$

**likewise** $\underline{P}(\{4\}) = 0 < \overline{P}(\{4\}) = 1/3;$

**however,** $P(\{4,6\}) = 1/3.$

**Moreover, the smallest algebra containing these 4 events is the power set of all 64 events on $\Omega$.**

**De Finetti's results apply also for conditional, 2-sided previsions, given an event F. These results obtain using called-off previsions of the form**

$$I_F\,\beta[X - P(X)]$$

**where $I_F$ is the indicator function for the conditioning event F.**
**Then, with a proviso, coherence assures that coherent called off 2-sided previsions are (finitely additive) conditional expectations, given the conditioning event.**

> **The restriction is that the conditioning event not be a null-event.**
> **Otherwise, a non-Archimedean (non-real valued) theory of previsions results.**

**When the random variables, $X_i$, include indicator functions for events, the resulting coherent 2-sided previsions include conditional probabilities for these events.**

- **Note well: The called-off previsions correspond only to *normal form*, and not *extensive form* decisions. There is no *dynamical coherence* in de Finetti's theory – merely *static* aspects of coherence are covered by his 2 theorems. The previous observation about the non-equivalence of *normal* and extensive form IP-decisions reinforces this boundary on the scope of de Finetti's *coherence*.**

- **Thus, de Finetti's theory of coherence does not require updating/learning by Bayesian conditional probabilities.**

**In order to link de Finetti's *coherence* with <u>I</u>P-theory where, the '<u>I</u>' stands for *<u>indeterminate</u>* previsions, we shift from 2-sided, to 1-sided previsions.**

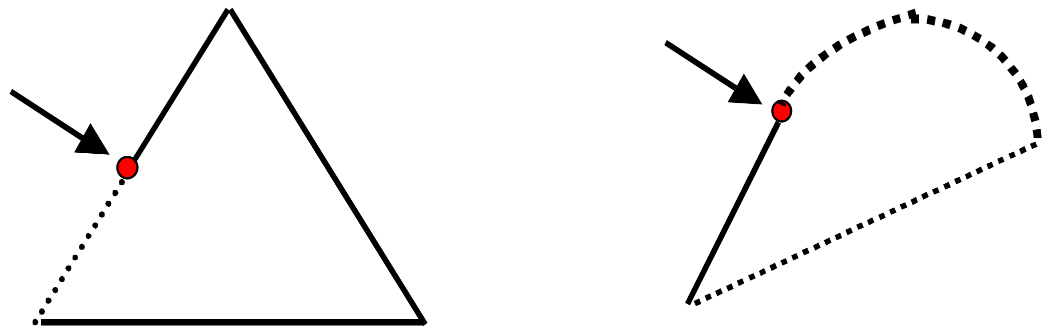
> **Then the decision maker is required only to provide a pair of (1-sided) previsions $\{\underline{P}(X_i), \overline{P}(X_i)\}$ for each random variable $X_i$ in $\chi$, corresponding to a largest "buy" price and smallest "sell" price for the corresponding 1-sided previsions, depending upon whether $\beta_I$ is positive, or negative, respectively.**

**Generalizations of de Finetti's *Coherence Theorem* for 1-sided previsions have been given by many researchers. There are variations, e.g., that use only closed intervals of previsions, and those with mixed boundaries for their IP sets of probabilities.**

**Since these generalizations of de Finetti's *coherence* all rely on *<u>binary comparisons</u>* between gambles, identifying those which are favorable versus others, the resulting IP sets of probabilities that may be distinguished from one another are *<u>convex</u>*, with relatively simple boundaries – where extreme points of the convex set are also exposed.**

- **The common technique uses one or another *Separating Hyperplane* Theorem.**

**Here are two convex sets with extreme points that are not exposed points.**
**(Dotted segments are open boundaries.)**



**Generalizations of de Finetti's *coherence* that use 1-sided previsions, those that persist in using binary comparisons exclusively to determine admissible options, in principle cannot distinguish such sets from rivals that differ on their boundaries.**

- **In higher dimensions, the dimension of the boundary may be large too!**

**Before concluding this selected review of aspects of de Finetti's *coherence* as that relates to decision making with IP sets, consider one more link between *strict dominance* and *coherence*, a link that pertains to de Finetti's insistence on allowing the *coherence* of (merely) finitely additive previsions.**

**For those who know my interests in finite additivity, it will come as a mild surprise that what comes next has little relation to non-conglomerability. It is *coherent* that for each $h_i$ in a denumerable partition $\{h_1, h_2, \ldots\}$, the conditional previsions satisfy:** $c_1 \leq P(X \mid h_i) \leq c_2$ **yet** $P(X) > c_2$.
**There is no contradiction with coherence, which requires that the decision maker responds to <u>finitely many</u> (called-off) previsions only at one time.**

**For what does come next, begin with an arbitrary set of real-valued, random variables, $\{X\}$, each defined on a common space, which we take as a σ-field of sets. It is enough to use the power set of a countable state space, $\Omega = \{\omega_i : i = 1, 2, \ldots\}$. So there are no issues of non-measurable sets.**

**For each variable, *X*, the decision maker has a prevision, P(*X*).**

- **When *X* is bounded, coherence entails that P(*X*) is real-valued.**
- **But when *X* is unbounded, P(*X*) *may* be infinite, negative or positive.**

- **When the prevision for $X$ is a real-value, it is 2-sided, with the real-value payoff $c_X[X - P(X)]$, where $c_X$ is a real number, either positive or negative. That is, the decision maker is committed to using $P(X)$ as the "fair price" when buying or selling the $|c_X|$-multiples of the random quantity $X$.**
- **When the prevision for $X$ is infinite-positive, i.e., when $X$ has a value to the decision maker greater than any finite amount, then for each real constant $k$ and for each $c_X > 0$, we require that the decision maker accepts (i.e., is committed to "buy") a 1-sided payoff $c_X(X - k)$.**

- **Likewise, when the prevision for $X$ is infinite-negative, with value less than any finite amount, then for each real constant $k$ and for $c_X < 0$, we require the decision maker accepts (i.e., is committed to "sell") a 1-sided payoff $c_X(X - k)$.**

- **In accord with deFinetti's theory, the decision maker is required to accept a *finite sum* of such real-valued payoffs across (finitely many) variables.**

***Definition* (as per de Finetti):**

**Previsions are *coherent* if there is no *finite* selection of non-zero constants, $c_X$, with the sum of the payoffs *uniformly* dominated by 0.**

**The previsions are *incoherent* otherwise.**

**The theory of coherent previsions is not confined to bounded random variables, as deFinetti notes (1974, Sections 3.12.4, 6.5.4-6.5.9).**

**However, in such cases, as previsions are not necessarily real-valued, they may induce a non-Archimedean ordering of random variables, as illustrated below with the St. Petersburg lottery.**

> ***Aside*: The von Neumann-Morgenstern Archimedean axiom requires that whenever there is a chain of strict preferences among 3 lotteries, $X \langle Y \langle Z$, there are compound lotteries, with $0 < \alpha, \beta < 1$, satisfying:**
>
> $$\alpha X \oplus (1-\alpha)Z \ \langle \ Y \ \langle \ \beta X \oplus (1-\beta)Z.$$

***Define* The (weak) order « over random variables according to their coherent previsions by:**

***Definition*: $X$ « $Y$ *if and only if* $P(Y - X) > 0$,**

**with $X \equiv Y$ *if and only if* $P(Y - X) = P(X - Y) = 0$.**

- **Under rather general conditions, illustrated below, the class of coherent weak-orders « with coherent previsions for unbounded random variables must distinguish by *strict preference* « among a finite set of *equivalent* (≈) random variables.**

**Two (real-valued) random variables are *equivalent* if they have the same probability distributions over *all* sets of outcomes.**

- **This situation is problematic for Expected Utility theory then, as strict preference is NOT a function of probability distributions over utilities!**
  **It is a challenge to any IP decision theory for unbounded variables that seeks to represent coherent preference as a function of sets of expectations**

**There are two cases worth distinguishing in this analysis:**

- **Non-Archimedean Previsions**
- **Discontinuous Previsions.**

- ***Non-Archimedean Previsions*: Let $Z$ be an unbounded variable so that for each coherent prevision, and for each real number $r > 0$, either $P(Z) > r$ or $P(Z) < -r$. Then, coherent prevision for $Z$ is infinite. An example of this is the familiar St. Petersburg variable, $Z$, where with probability $2^{-n}$, $Z = 2^n$. Flip a fair coin until it lands heads first at flip $n$, when $Z$ equals $2^n$.**
- **A non-Archimedean ordering of random variables results,**

**The von Neumann-Morgenstern Archimedean axiom requires that whenever there is a chain of strict preferences among 3 lotteries, $X \langle Y \langle Z$, there are compound lotteries, with real-valued $0 < \alpha, \beta < 1$, satisfying:**

$$\alpha X \oplus (1-\alpha)Z \;\langle\; Y \;\langle\; \beta X \oplus (1-\beta)Z.$$

**When $X = 1$, $Y = 2$, and $Z = St.P$, there is no real-valued $0 < \alpha < 1$ satisfying**

$$\alpha X \oplus (1-\alpha)Z \;\langle\; Y = 2$$

- ***Discontinous, coherent previsions*.**

**Let $\{X_n\}$ ($n = 1, \ldots$) be a sequence of non-negative random variables that converges (pointwise, *from below*) to the random variable $X$.**

***Definition*: Say that previsions are <u>*continuous*</u> if $lim_n\, P(X_n) = P(X)$.**

- **When random variables are bounded, as is well known (deFinetti), coherent previsions are continuous *if and only if* they are the expectations of some countably additive probability.**

- **But when random variables are unbounded, they may fail to be continuous, even when probabilities are countably additive.**

***Previsions that differ from their expected values*:**

**Let *Z* be an unbounded, discrete random variable (*Z* = 1, 2, …).**

***Example: Z*  is Geometric(p):   Prob(Z = *n*) = $p(1-p)^{n-1}$   (*n* = 1, 2, …).**

- **Coherence assures that  P(Z) ≥ E[Z] the expected value of Z.**

**Let the prevision for *Z*, P(*Z*), be <u>greater than</u> its expectation, E[*Z*], where for the Geometric(p), E[*Z*] = $p^{-1}$. So, then we can write P(*Z*) = E[*Z*] + *b*, with *b* > 0.**

- **Let '*b*' denote the *boost* that the prevision receives in excess of its expectation.**

**Possibly, *b* is +∞ in the example, though assume not in what follows.**

**BUT P(*Z*) = E[Z] + *b* (*b* ≥ 0) is a coherent prevision!**

**If *b* is finite, the prevision is a 2-sided coherent prevision in deFinetti's sense, since there can be no *sure loss* when this prevision is combined with coherent previsions for bounded variables.**

- **But this prevision is <u>not</u> *continuous* when *b* > 0.**

**For each *n* (*n* = 1, 2, …) define the bounded random variable $Z_n$ by**
**$Z_n$ = 0 for states when *Z* > *n*, and $Z_n$ = *Z* otherwise.**
**So, P(Z) = E(Z), since the can be no "boost" for bounded variables.**
**The sequence {$Z_n$} converges from below to Z.**

**Of course, since the probability is countably additive, $lim_n$ E[$Z_n$] = E[*Z*].**
**However, P(*Z*) = E[*Z*] + *b*.**

- **So, *when b* > 0, previsions are *not continuous*.**

**Strict preference among equivalent random variables.**

***Theorem 1*: Let *X* be a Geometric(p) random variable. Assume the prevision for *X*, P(*X*), is finite but greater than its expectation (E[*X*] = $p^{-1}$),**

**then there exist three equivalent random variables, $W_1$, $W_2$, and $W_3$, each with finite prevision, such that no two have the same prevision.**

***Example*: The following illustrates the theorem for the special case of fair-coin flipping, e.g., p = ½ , Prob(*X* = *n*) = $2^{-n}$, *n* = 1, 2, …, with E[*X*] = 2.**

**Let {*B*, $B^c$} be the outcome of an independent flip of another fair coin, so that Prob(*B*, *X* = *n*) = $2^{-(n+1)}$ for *n* = 1, 2, … .**

**Define $W_1$ and $W_2$ by:**

| | $\omega_1$ | $\omega_2$ | .... | $\omega_n$ | .... |
|---|---|---|---|---|---|
| | $X = 1$ | $X = 2$ | | $X = n$ | |
| **B** | $W_1 = 2$ | $W_1 = 3$ | | $W_1 = n+1$ | |
| | $W_2 = 1$ | $W_2 = 1$ | | $W_2 = 1$ | |
| | $X = 1$ | $X = 2$ | | $X = n$ | |
| **B$^c$** | $W_1 = 1$ | $W_1 = 1$ | | $W_1 = 1$ | |
| | $W_2 = 2$ | $W_2 = 3$ | | $W_2 = n+1$ | |

**That is:**

$$W_1(B, X = n) = n+1; \;\; W_1(B^c, X = n) = 1 \quad (n = 1, 2, ...)$$

**and** $$W_2(B^c, X = n) = n+1; \;\; W_2(B, X = n) = 1 \quad (n = 1, 2, ...).$$

- **Obviously, $W_1$ and $W_2$ are equivalent.**
- **Each has a Geometric(½) distribution;**
- **Hence, $X \approx W_1 \approx W_2$.**
- **However,** $W_1 + W_2 - X = 2$.

**Thus,** $P(W_1 - X) + P(W_2 - X) = 0$ ***if and only if***

$$P(W_1) = P(W_2) = P(X) = 2,$$

**when the prevision for a Geometric(½) variable is its expectation, and previsions are continuous, with no *boost*: $b = 0$.**

***Non-Archimedean preferences for generalized St. Petersburg variables.***

**We turn next to non-Archimedean previsions that result from random variables whose coherent prevision is mandated to be infinite, as with the St. Petersburg lottery.**

**Consider the class of Geometric($p_m$) distributions, $p_m = 1 - 2^{-m}$, $m = 2, 3, \ldots$ .**

**For each member of this class, we define a generalized St. Petersburg gamble, $Z_m$, and construct a set of $2^{m-1}$ equivalent random variables $X_1 \approx X_2 \approx \ldots \approx X_{2^{m-1}}$ ($\approx Z_m$), so that coherent previsions, though infinite, obey:**

***Theorem 2*:** $$\sum_{i=1}^{2^{m-1}} [P(X_i) - P(Z_m)] = 2^{m-1}. \qquad (*)$$

**That is, though each of $X_i$ ($i = 1, \ldots, 2^{m-1}$) and $Z_m$ are pairwise equivalent variables, the coherent prevision of their differences cannot all be 0.**

***Example*: We illustrate this result for a special case, the Geometric(½). The construction parallels the illustration of *Theorem 1*.**

***The St. Petersburg gamble***

- **Let states $\omega_n$ ($n$ = 1, ….) carry the Geometric(½) distribution,**

  **Prob($\omega_n$) = $2^{-n}$ ($n$ = 1, 2, …).**

- **Let {$B$, $B^c$} be another fair-coin flip, independent of the states, $\omega_n$.**

**As before, partition each $\omega_n$ into two equi-probable cells using {$B$, $B^c$} and define three (equivalent) random variables, $X$, $Y$, and $Z$, as follows:**

| | $\omega_1$ | $\omega_2$ | .... | $\omega_n$ | .... |
|---|---|---|---|---|---|
| | $Z = 2$ | $Z = 4$ | | $Z = 2^n$ | |
| **B** | $X_1 = 4$ | $X_1 = 8$ | | $X_1 = 2^{n+1}$ | |
| | $X_2 = 2$ | $X_2 = 2$ | | $X_2 = 2$ | |
| | $Z = 2$ | $Z = 4$ | | $Z = 2^n$ | |
| **B$^c$** | $X_1 = 2$ | $X_1 = 2$ | | $X_1 = 2$ | |
| | $X_2 = 4$ | $X_2 = 8$ | | $X_2 = 2^{n+1}$ | |

- **the St. Petersburg random variable: $Z(\omega_n) = 2^n$, independent of $B$.**
- **the random variable $X_1(\omega_n \cap B) = 2^{n+1}$, and $X_1(\omega_n \cap B^c) = 2$.**
- **the random variable $X_2(\omega_n \cap B) = 2$, and $X_2(\omega_n \cap B^c) = 2^{n+1}$.**

**Though** $X_1 \approx X_2 \approx Z$,

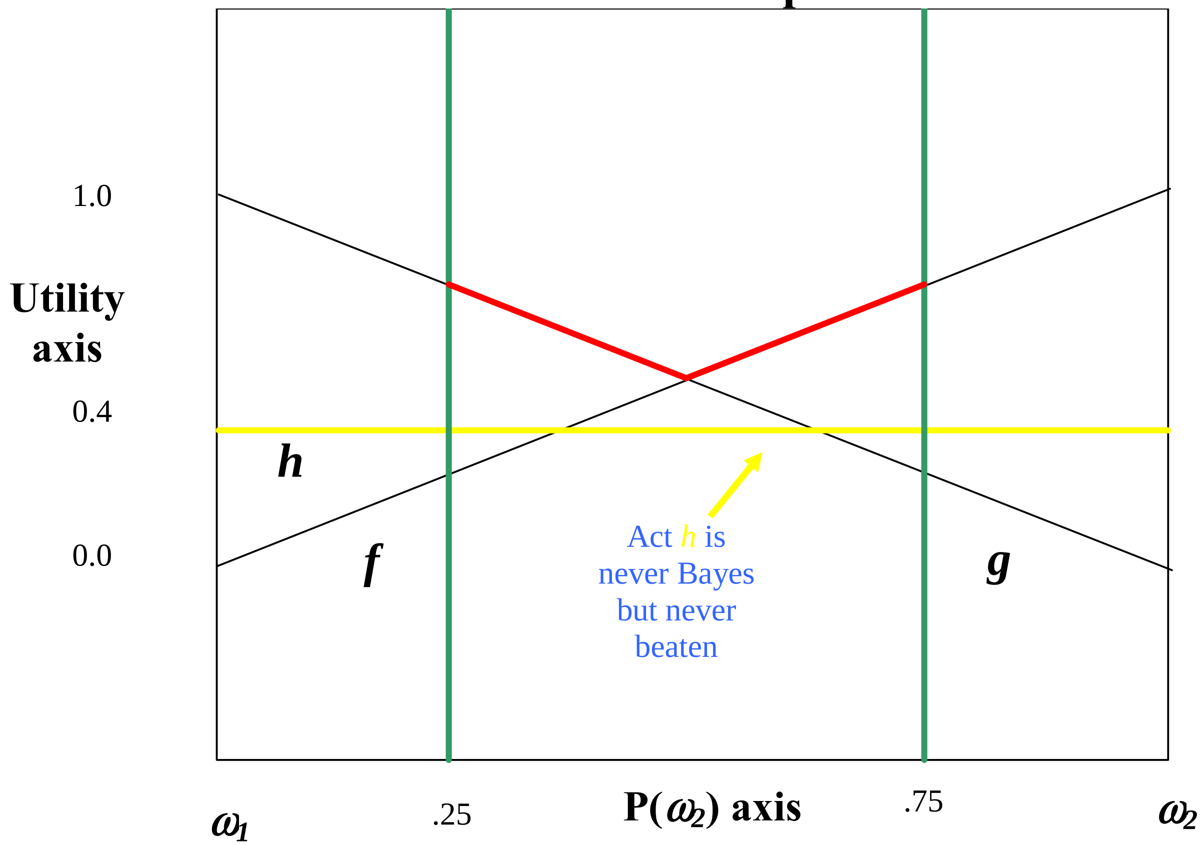
**if previsions are coherent,** $P(X_1 + X_2 - 2Z) = 2$.

**This is in contradiction with the hypothesis that the prevision of the difference between equivalent random variables is 0, as that requires**

$$P(X_1 - Z) + P(X_2 - Z) = P(X_1 + X_2 - 2Z) = 0.$$

- **Thus, de Finetti's *coherence* of previsions, respect for uniform, strict dominance over a (countable) partition, precludes preference according to SEU, when discontinuous or non-Archimedean previsions obtain and equivalent variables cannot be given equal previsions.**

**Return to question the relation between IP-decision theory and simple dominance**
**Reconsider this decision problem.**




**Only {*f*,*g*} are Bayes-admissible from the triple {*f*,*g*,*h*}; however, all pairs are Bayes-admissible in pairwise choices. Levi calls *h* <u>*second worst*</u> in the triple {*f*,*g*,*h*}.**

**Contrast three *coherent* decision rules for extending Expected Utility [*EU*] theory when probability – but not cardinal utility – is indeterminate.**

**The decision problems involve (bounded) sets of lotteries, where the outcomes have well-defined cardinal utility but where the (act-independent) states are uncertain, represented by a *convex* set of probabilities $\mathcal{P}$.**

- ***Γ-Maximin*** **(Gilboa-Schmeidler) – maximize minimum expectations over $\mathcal{P}$.**
- ***Maximality*** **(Walley) – admissible choices are undominated in expectations over over $\mathcal{P}$ by any single alternative choice.**
- ***E-admissibility*** **(Levi/Savage) – admissible choices have Bayes' models, i.e., they maximize *EU* for some probability in the (convex) set $\mathcal{P}$.**

**Each rule has *EU* Theory as a special case when probability is determinate, i.e., when $\mathcal{P}$ is comprised by a single probability distribution.**

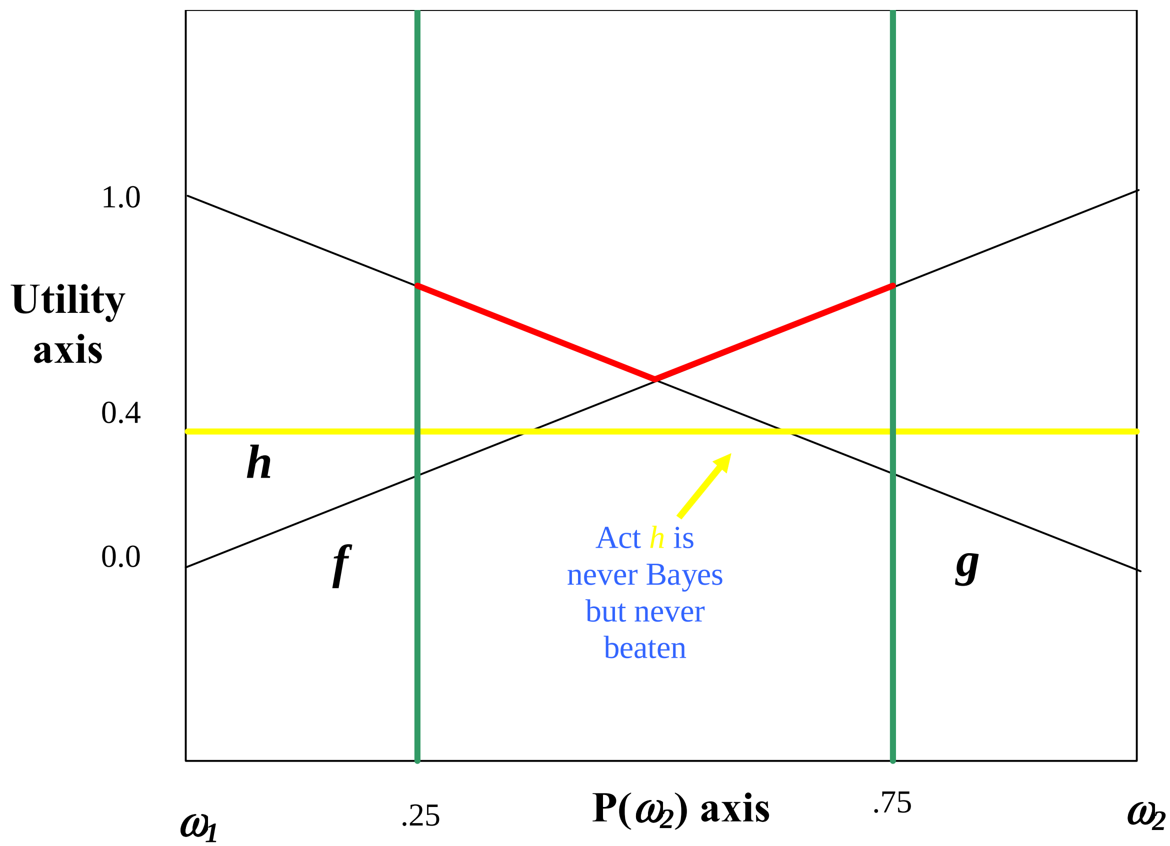
**And each rule is *coherent* in the sense that sure loss (*Book*) is not possible.**

**The three rules are chosen to reflect the following progression, where each rule relaxes more of the ordering assumption than does its predecessors:**

- ***Γ-Maximin*** **produces a (real-valued) ordering of options; hence, defined by binary comparisons – but it fails *Independence*.**

- ***Maximality*** **does not generate an ordering of options; however, it is given by binary comparisons.**

- ***E-admissibility*** **does not generate an ordering, nor is it given by binary comparisons.**



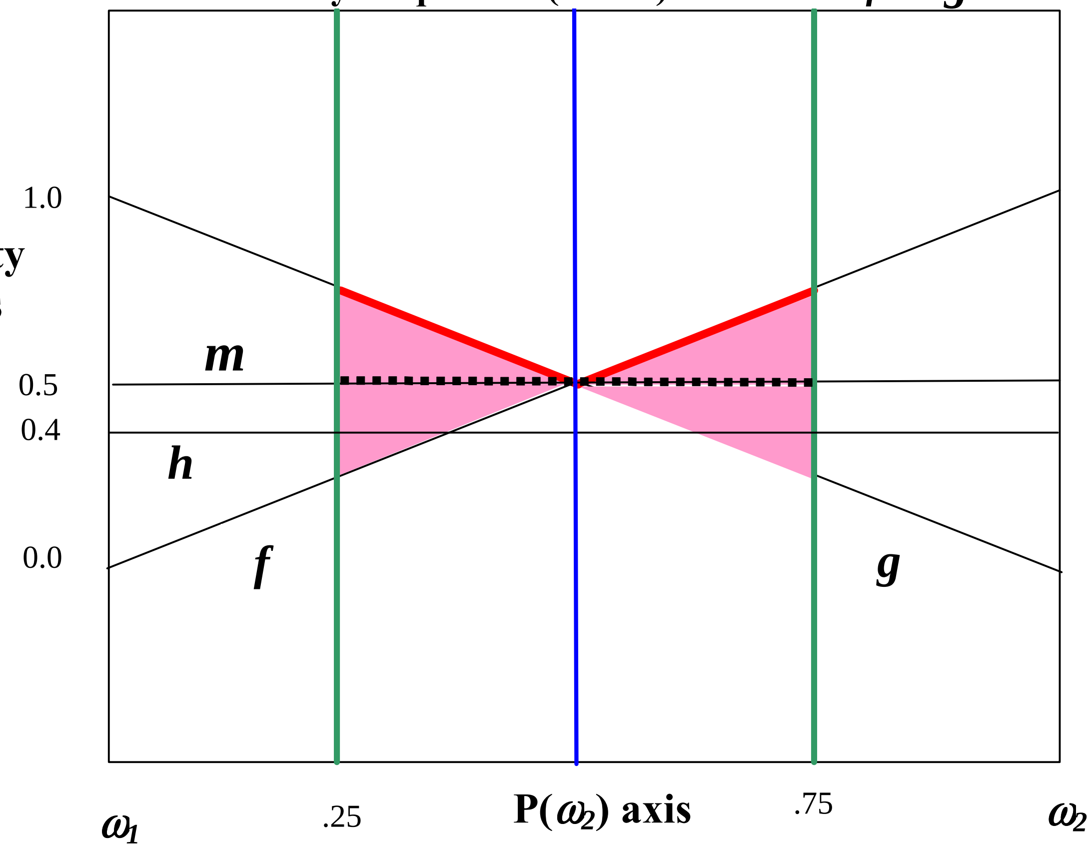

- ***The Γ-Maximin* solution is {*h*}.**
- **The *E-admissible* solution set is {*f*, *g*}.**
- **And *Maximality* finds all three options admissible, {*f*, *g*, *h*}.**

**Thus, each rule gives a different set of admissible options in this problem.**

**Create a convex option space by allowing mixed strategies.**

**Expected Utility for the (Bayes) mixed options $\alpha f \oplus (1-\alpha)g$ is in pink; they maximize EU at $p(\omega_2) = .5$ (blue)**

**The Bayes equalizer (mixed) act is $m = .5f \oplus .5g$**




- ***The Γ-Maximin*** **solution is the EU-*equalizer* {*m*}.**
- **The *E-admissible* and *Maximally admissible* options are the same set of *Bayes* solutions (pink).**

**The agreement of the 3 decision rules on Bayes solutions is no accident as:**

- ***Walley* (Theorem 3.9.5, 1990) establishes that when the option set is *convex* and the (convex) set of probabilities $\mathcal{P}$ is *closed*, then**

  ***E-admissibility* and *Maximality* give the same solution sets:**

  **Their admissible sets are precisely the Bayes-admissible options.**

- **And then it also follows that the *Γ-Maximin* admissible acts are a (proper) subset of the Bayes-admissible options.**

***Under these conditions, pairwise comparisons of acts suffice to determine the set of Bayes-admissible choices.***

**D.Pearce (1984), reports a related result which is important for understanding the underlying connection between *dominance* and *Bayes-admissibility*.**

***Theorem* (Pearce, 1984): In a decision problem under uncertainty,**

- **with finitely many states and finitely generated option set *O*,**
- **with utility of outcomes determinate,**

**if an option $o \in O$ fails to be Bayes-admissible,**

**then *o* is uniformly, strictly dominated by a finite mixture from *O*.**

**Aside: This result can be extended to infinite decision problems.**

**In this sense, incoherent choices suffer deFinetti's penalty – being uniformly strictly dominated by a mixed option – within the decision at hand – and not merely for previsions, which are specialized decisions.**

**In accord with Pearce's Theorem, in the example above, the mixed act $m = .5f \oplus .5g$ strictly dominates *h*.**

**I conclude by illustrating how this insight relates to coherent choice in the setting of games, which is the domain where Pearce directs his analysis,**

**Game 1: *Iterative elimination of dominated strategies*.**
**Consider the following $2 \times 3$ game, with utility payoffs (*row*, *column*).**

| | ***L*** | ***M*** | ***R*** |
|---|---|---|---|
| ***T*** | **(2,1)** | **(1,4)** | **(0,3)** |
| ***B*** | **(1,8)** | **(0,2)** | **(1,3)** |

**There is no strict dominance between Row-player's two options.**
**But from Column-player's perspective, *R* is *never* a best reply.**
**For example, the mixed strategy [.25*L* $\oplus$ .75*M*] strictly dominates *R*.**

**So, if Row-player models Column player as one who plays Bayes-admissible options, from Row-player's perspective, state *R* may be eliminated. Then *T* dominates *B*. And if Column-player models Row-player also as playing Bayes-admissible options (and models Column-player likewise), then Column-player may eliminate state *B*. This results in the choices (*T*, *M*), which also is the unique Nash equilibrium of this game.**

***Game 2*: *An undominated option that is not part of a Nash equilibrium***

**Consider the following 2×2×4, 3-player game: where player-1 chooses a Row, player-2 chooses a Column, and player 3 chooses a Matrix {*A*,*B*,*C*,*D*}.**

**The numbers are the utility payoffs to player 3.**

**A**

| | *L* | *R* |
|---|---|---|
| *T* | 9 | 0 |
| *B* | 0 | 0 |

**B**

| | *L* | *R* |
|---|---|---|
| *T* | 0 | 9 |
| *B* | 9 | 0 |

**C**

| | *L* | *R* |
|---|---|---|
| *T* | 0 | 0 |
| *B* | 0 | 9 |

**D**

| | *L* | *R* |
|---|---|---|
| *T* | 6 | 0 |
| *B* | 0 | 6 |

**Matrix *D* is undominated by any mixture of {*A*,*B*,*C*}; however, it is not part of a Nash equilibrium. This is because <u>*all*</u> of *D*'s Bayes-models live <u>*off*</u> the surface of independence between Row "states" and Column "states," which is where all Nash-equilibria strategies must reside! *D* is not a *best reply* to a probability on $R \times C$ that makes *row-states* independent of *column-states*. Why does rationality in games require such independence? (It is not a matter of act/state independence.)**

- **When we apply IP-decision making to non-cooperative games, then *coherence* – avoiding dominated options – does not result in an endorsement of Nash's criterion of equilibria! What results is the theory of *Rationalizable* strategies.**

## Summary

1. **Review of structural (not normative) assumptions for this overview.**
   - **Moral hazards and dominance**
   - **State-dependent utilities**
   - **Normal versus extensive form decision making**

2. **IP theory relating to de Finetti's *coherence of previsions*.**
   - ***Fundamental Theorem*: *imprecise* vs. *indeterminate* previsions**
   - **Limitations of binary comparisons with *coherent* IP-sets**
   - **Dominance and coherent previsions for unbounded variables**

3. **IP theories related to relaxing canonical axioms of SEU theory**
   - **Γ-*Maximin*, *Maximality*, and *E-Admissibility***
   - **On dominance and rationalizability-- D.Pearce's (1984) result.**